DEFINITION: Let F be a field, V an F-vector space of dimension n, and $\phi : V \to V$ a linear transformation.

- The **characteristic polynomial** of ϕ is the polynomial $c_\phi(x) := \det(xI_n - [\phi]_B^B)$ for a basis B of V.
- The **minimal polynomial** of ϕ is the monic generator $m_\phi(x)$ of the ideal $\operatorname{ann}_{F[x]}(V_\phi)$. Equivalently, $m_\phi(x)$ is the monic polynomial of smallest degree such that $m_\phi(\phi) = 0$.

We write $c_A(x) := c_{t_A}(x)$ and $m_A(x) := m_{t_A}(x)$ for a matrix A.

PROPOSITION: Let F be a field, V an F-vector space of dimension n, and $\phi : V \to V$ a linear transformation. Let $g_1 \mid \cdots \mid g_k$ be the invariant factors of ϕ. Then,

(1) $m_\phi(x) = g_k$.
(2) $c_\phi(x) = g_1 \cdots g_k$.

COROLLARY (CAYLEY-HAMILTON): $m_\phi(x) \mid c_\phi(x)$.

THEOREM: Let F be a field, V an F-vector space of dimension n, and $\phi : V \to V$ a linear transformation. For $\lambda \in F$, λ is an eigenvalue of $\phi \iff m_\phi(\lambda) = 0 \iff c_\phi(\lambda) = 0$.

(1) Let $A = \begin{bmatrix} -11 & -4 & -2 \\ 18 & 7 & 3 \\ 18 & 6 & 4 \end{bmatrix} \in \mathrm{Mat}_3(\mathbb{Q})$. The SNF of $xI - A \in \mathrm{Mat}_3(\mathbb{Q}[x])$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x^2+x-2 \end{bmatrix}$.

(a) What is the minimal polynomial of A?
(b) What is the characteristic polynomial of A?
(c) What is the RCF of A?
(d) What are the eigenvalues of A?

(a) $x^2 + x - 2$
(b) $(x-1)(x^2+x-2)$
(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$
(d) $1, -2$

(2) Let F be a field, and $A \in \mathrm{Mat}_n(F)$ be a nilpotent matrix, meaning that $A^t = 0$ for some $t \geq 1$.

(a) What can you deduce about $m_A(x)$ from the fact that $A^t = 0$?
(b) Prove that $A^n = 0$.
(c) If $n = 4$, what are the possible lists of invariant factors?
(d) Give a complete nonredudant list of representatives of similarity classes of 4×4 nilpotent matrices.

(a) $m_A(x) \mid x^t$.
(b) From part (a), $m_A(t)$ is a power of x. Since the degree of $m_A(x)$ is at most n, $m_A(x) = x^j$ for some $j \leq n$. Then x^n is a multiple of $m_A(x)$, so $A^n = 0$.
(c) Each invariant factor is a power of x dividing the next. We can have x^4, $x|x^3$, $x^2|x^2$, $x|x|x^2$ or $x|x|x|x$.
(d) We take the RCF of each list above:

$$\begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

(3) Let $f(x) = (x^2-1)(x^4-1) \in \mathbb{Q}[x]$.

(a) If $A \in \mathrm{Mat}_6(\mathbb{Q})$ has characteristic polynomial $c_A(x) = f(x)$, then what are the possible lists of invariant factors of A?

(b) Give a complete nonredudant list of representatives of similarity classes of rational matrices with characteristic polynomial f.

> **(a)** We must have that the product of the invariant factors is $f(x)$, and each one divides the next. The possibilities are $(x^2-1)(x^4-1)$, $x-1|(x+1)(x^4-1)$, $x+1|(x-1)(x^4-1)$, $x^2-1|x^4-1$.
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> **(b)** We take the RCF of each list above:
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> $$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(4) Proofs:

(a) Show that the characteristic polynomial is well-defined (i.e., independent of choice of B).

(b) Prove[1] the Proposition and deduce Cayley-Hamilton as a Corollary.

(c) Prove the Theorem.

> **(a)** Set $A = [\phi]_B^B$. Then for any other basis, the matrix A' of ϕ is PAP^{-1} for some P. Then $xI - A' = P(xI-A)P^{-1}$, so $\det(xI-A') = \det(P)\det(xI-A)\det(P^{-1}) = \det(xI-A)$.
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> **(b)** We showed in the Homework for a diagonal matrix with entries dividing each other, if there is a nonzero entry in each row, then the annihilator of the module presented by that matrix is generated by the last entry, Part (1) is a special case of this. Part (2) follows from the formula for the Smith Normal Form in terms of determinants.
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> **(c)** Since $m_\phi | c_\phi$, any irreducible factor of m_ϕ is also an irreducible factor of c_ϕ. Conversely, given an irreducible factor of c_ϕ, it must be an irreducible factor of g_i for some i since $F[x]$ is a UFD, and since $g_i \mid m_\phi$, it must be an irreducible factor of m_ϕ. Thus, the irreducible factors of m_ϕ and c_ϕ are the same. In particular, $(x-\lambda)|m_\phi$ if and only if $(x-\lambda)|c_\phi$, so λ is a root of m_ϕ if and only if it is a root of c_ϕ. Now, if λ is a root of c_ϕ, then $\det(\lambda I - A) = 0$, so $\ker(A-\lambda I) \neq 0$. A nonzero vector $v \in \ker(A - \lambda I)$ satisfies $Av = \lambda v$, and thus is an eigenvector. The converse is roughly the same.

(5) Find the minimal and characteristic polynomials of rotation by $\pi/3$ counterclockwise in $\mathbb{R}^2$.

(6) Let F be a field.

(a) Let A and B be two 3×3 matrices with entries in F. Prove A and B are similar if and only if they have the same characteristic polynomial and the same minimal polynomial.

(b) Show, by way of an example with justification, that the statement in part (a) would become false if 3×3 were replaced by 4×4.

(7) Give a complete nonredundant list of representatives for the conjugacy classes of $\mathrm{GL}_3(\mathbb{Z}/2)$.

[1] Hint: You did most of the work for (1) in a homework problem.