

DEFINITION: Let  $F$  be a field,  $V$  an  $F$ -vector space of dimension  $n$ , and  $\phi : V \rightarrow V$  a linear transformation.

- The **characteristic polynomial** of  $\phi$  is the polynomial  $c_\phi(x) := \det(xI_n - [\phi]_B^B)$  for a basis  $B$  of  $V$ .
- The **minimal polynomial** of  $\phi$  is the monic generator  $m_\phi(x)$  of the ideal  $\text{ann}_{F[x]}(V_\phi)$ . Equivalently,  $m_\phi(x)$  is the monic polynomial of smallest degree such that  $m_\phi(\phi) = 0$ .

We write  $c_A(x) := c_{t_A}(x)$  and  $m_A(x) := m_{t_A}(x)$  for a matrix  $A$ .

PROPOSITION: Let  $F$  be a field,  $V$  an  $F$ -vector space of dimension  $n$ , and  $\phi : V \rightarrow V$  a linear transformation. Let  $g_1 \mid \cdots \mid g_k$  be the invariant factors of  $\phi$ . Then,

- (1)  $m_\phi(x) = g_k$ .
- (2)  $c_\phi(x) = g_1 \cdots g_k$ .

COROLLARY (CAYLEY-HAMILTON):  $m_\phi(x) \mid c_\phi(x)$ .

THEOREM: Let  $F$  be a field,  $V$  an  $F$ -vector space of dimension  $n$ , and  $\phi : V \rightarrow V$  a linear transformation. For  $\lambda \in F$ ,  $\lambda$  is an eigenvalue of  $\phi \iff m_\phi(\lambda) = 0 \iff c_\phi(\lambda) = 0$ .

(1) Let  $A = \begin{bmatrix} -11 & -4 & -2 \\ 18 & 7 & 3 \\ 18 & 6 & 4 \end{bmatrix} \in \text{Mat}_3(\mathbb{Q})$ . The SNF of  $xI - A \in \text{Mat}_3(\mathbb{Q}[x])$  is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x^2+x-2 \end{bmatrix}$ .

- (a) What is the minimal polynomial of  $A$ ?
- (b) What is the characteristic polynomial of  $A$ ?
- (c) What is the RCF of  $A$ ?
- (d) What are the eigenvalues of  $A$ ?

(2) Let  $F$  be a field, and  $A \in \text{Mat}_n(F)$  be a nilpotent matrix, meaning that  $A^t = 0$  for some  $t \geq 1$ .

- (a) What can you deduce about  $m_A(x)$  from the fact that  $A^t = 0$ ?
- (b) Prove that  $A^n = 0$ .
- (c) If  $n = 4$ , what are the possible lists of invariant factors?
- (d) Give a complete nonredundant list of representatives of similarity classes of  $4 \times 4$  nilpotent matrices.

(3) Let  $f(x) = (x^2 - 1)(x^4 - 1) \in \mathbb{Q}[x]$ .

- (a) If  $A \in \text{Mat}_6(\mathbb{Q})$  has characteristic polynomial  $c_A(x) = f(x)$ , then what are the possible lists of invariant factors of  $A$ ?
- (b) Give a complete nonredundant list of representatives of similarity classes of rational matrices with characteristic polynomial  $f$ .

(4) Proofs:

- (a) Show that the characteristic polynomial is well-defined (i.e., independent of choice of  $B$ ).
- (b) Prove<sup>1</sup> the Proposition and deduce Cayley-Hamilton as a Corollary.
- (c) Prove the Theorem.

(5) Find the minimal and characteristic polynomials of rotation by  $\pi/3$  counterclockwise in  $\mathbb{R}^2$ .

(6) Let  $F$  be a field.

- (a) Let  $A$  and  $B$  be two  $3 \times 3$  matrices with entries in  $F$ . Prove  $A$  and  $B$  are similar if and only if they have the same characteristic polynomial and the same minimal polynomial.
- (b) Show, by way of an example with justification, that the statement in part (a) would become false if  $3 \times 3$  were replaced by  $4 \times 4$ .

(7) Give a complete nonredundant list of representatives for the conjugacy classes of  $\text{GL}_3(\mathbb{Z}/2)$ .

<sup>1</sup>Hint: You did most of the work for (1) in a homework problem.