DEFINITION: Let R be a commutative ring, and $A \in \mathrm{Mat}_{n\times n}(R)$. The **determinant** of A is

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)}.$$

THEOREM 1: Identify $\mathrm{Mat}_{n\times n}(R)$ with $\underbrace{R^n \times \cdots \times R^n}_{n \text{ times}}$ by considering a matrix as an n-tuple of columns. The determinant is the unique function

$$\det \colon \underbrace{R^n \times \cdots \times R^n}_{n\text{times}} \to R$$

that satisfies the following three properties:

- det is **multilinear**, meaning

$$\begin{aligned}\det(v_1, \dots, v_{i-1}, \boldsymbol{v} + \boldsymbol{w}, v_{i+1}, \dots, v_n) &= \det(v_1, \dots, v_{i-1}, \boldsymbol{v}, v_{i+1}, \dots, v_n) \\ &\quad + \det(v_1, \dots, v_{i-1}, \boldsymbol{w}, v_{i+1}, \dots, v_n) \\ \det(v_1, \dots, v_{i-1}, \boldsymbol{rv}, v_{i+1}, \dots, v_n) &= \boldsymbol{r} \det(v_1, \dots, v_{i-1}, \boldsymbol{v}, v_{i+1}, \dots, v_n)\end{aligned}$$

- det is **alternating**, meaning

$$\det(v_1, \dots, v_n) = 0 \qquad \text{if } v_i = v_j \text{ for some } i \neq j.$$

- $\det(e_1, \dots, e_n) = 1$.

(1) Working with Theorem 1:

(a) Use Theorem 1 to explain why the determinant of a diagonal matrix is the product of its diagonal entries.

(b) Use Theorem 1 to show that if some column of A is a linear combination of the other columns of A, then $\det(A) = 0$.

(c) Use part (b) to show that if $R = F$ is a field, and A is not invertible, then $\det(A) = 0$.

(d) Use Theorem 1 to show[1] that

$$\det(\boldsymbol{v_2}, \boldsymbol{v_1}, v_3, \dots, v_n) = -\det(\boldsymbol{v_1}, \boldsymbol{v_2}, v_3, \dots, v_n).$$

Likewise, the same holds for swapping any two entries.

(a) We have

$$\det \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix} = \det(r_1 e_1, r_2 e_2, \dots, r_n e_n) = r_1 r_2 \cdots r_n \det(e_1, e_2, \dots, e_n) = r_1 r_2 \cdots r_n.$$

(b) Say $v_1 = \sum_{j>1} r_j v_j$; then

$$\det(\sum_{j>1} r_j v_j, v_2, \dots, v_n) = \sum_{j>1} r_j \det(v_j, v_2, \dots, v_n) = 0.$$

[1]Hint: Consider $\det(\boldsymbol{v_1} + \boldsymbol{v_2}, \boldsymbol{v_1} + \boldsymbol{v_2}, v_3, \dots, v_n)$.

A similar argument applies when another entry is a sum of the others.

(c) Note that if one column is a linear combination of the others that the columns are linearly dependent, and conversely, if the columns are linearly dependent, since F is a field, one can solve for one of the columns as a linear combination of the others. Now we claim that if the columns of A are linearly independent, then A is invertible. Indeed, if the columns are linearly independent then the kernel of the linear transformation $t_A : F^n \to F^n$ of multiplication by A is zero (since a vector in the null space gives a dependence relation on the columns), and the dimension of the image is n by Rank-Nullity, so t_A is bijective, and hence an isomorphism. Thus A has an inverse, given by the matrix of the inverse of t_A in the standard bases.

Thus, if A is not invertible, some columns of A is a linear combination of the others, and $\det(A) = 0$ by part (b).

(d) We have

$$\begin{aligned}0 &= \det(v_1 + v_2, v_1 + v_2, \dots) = \det(v_1 + v_2, v_1, \dots) + \det(v_1 + v_2, v_2, \dots)\\ &= \det(v_1, v_1, \dots) + \det(v_1, v_2, \dots) + \det(v_2, v_1, \dots) + \det(v_2, v_2, \dots)\\ &= \det(v_1, v_2, \dots) + \det(v_2, v_1, \dots)\end{aligned}$$

and the claim follows.

(2) Uniqueness part of Theorem 1:

(a) Use 1(d) to show that for any $\sigma \in S_n$,

$$\det(v_{\boldsymbol{\sigma(1)}}, \dots, v_{\boldsymbol{\sigma(n)}}) = \operatorname{sgn}(\sigma) \det(v_1, \dots, v_n).$$

(b) Explain the following claim: if $F\colon \underbrace{R^n \times \cdots \times R^n}_{n\text{times}} \to R$ is multilinear, then F is completely determined by $F(e_{i_1}, \dots, e_{i_n})$ for $1 \le i_1, \dots, i_n \le n$.

(c) Explain the following claim: if $F\colon \underbrace{R^n \times \cdots \times R^n}_{n\text{times}} \to R$ is multilinear and alternating, then F is completely determined by $F(e_1, \dots, e_n)$.

THEOREM 2: Let R be a commutative ring and $A, B \in \mathrm{Mat}_{n\times n}(R)$. Then
$$\det(AB) = \det(A)\det(B).$$

PROPOSITION: Let R be a commutative ring. Let A be a square matrix, and B be a matrix obtained from A by an elementary column operation.

- For the operation "add $r \in R$ times column i to column j" we have $\det(B) = \det(A)$.
- For the operation "multiply column i by $u \in R^\times$" we have $\det(B) = u\det(A)$.
- For the operation "swap column i and column j" we have $\det(B) = -\det(A)$.

(3) Use Theorem 1 to prove the Proposition.

Write $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$.

Say that B is obtained from A by adding r times column 1 to column 2. Then $B = \begin{bmatrix} v_1 & v_2 + rv_1 & \cdots & v_n \end{bmatrix}$ and
$$\det(B) = \det(\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}) + r\det(\begin{bmatrix} v_1 & v_1 & \cdots & v_n \end{bmatrix}) = \det(A).$$
Similarly for i and j in place of 1 and 2.

Say that B is obtained from A by multiplying column 1 by u. Then $B = \begin{bmatrix} uv_1 & v_2 & \cdots & v_n \end{bmatrix}$ and
$$\det(B) = \det(\begin{bmatrix} uv_1 & v_2 & \cdots & v_n \end{bmatrix}) = u\det(A).$$
Similarly for general i.

The column swap operation was 1(d).

(4) Use the Proposition (and not the definition) to compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 7 & 13 \end{bmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{Q})$.

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(5) Proof of Theorem 2 in the case $R = F$ is a field:

(a) Prove Theorem 2 in the case $B = E$ is an elementary matrix.

(b) Prove[2] Theorem 2 in the case A and B are both invertible matrices.

(c) Show that AB is invertible if and only if A and B are both invertible.

(d) Show[3] that $\det(A) \in F^\times$ if and only if A is invertible.

(e) Complete the proof of Theorem 2 in the field case.

(a) Recall that AE is the matrix obtained from A by an elementary column operation of the same "type". Moreover, E is the matrix obtained from the identity by the same elementary column operation. Thus, when E is type I, we have $\det(E) = 1$ and $\det(AE) = \det(A)$ by the Proposition, so the claim holds. We check type II and type III in the same way.

[2]Hint: You can use the fact that over a field, every invertible matrix is a product of elementary matrices

[3]Hint: Use part (a) and 1(c).

(b) Let $B = E_1 \dots E_n$. By part (a) and induction on n, we have $\det(B) = \det(E_1) \cdots \det(E_n)$, and also by part (a) and induction on n, we have $\det(AB) = \det(A)\det(E_1) \cdots \det(E_n)$, so $\det(AB) = \det(A)\det(B)$.

(c) If A and B are invertible, then $(AB)B^{-1}A^{-1} = B^{-1}A^{-1}(AB) = I$. If AB is invertible, then $A(B(AB)^{-1}) = I$ implies that t_A is surjective, and hence bijective by a Rank-Nullity argument akin to 1(c). Similarly, $((AB)^{-1}A)B = I$ implies that t_B is injective, and hence surjective by a Rank-Nullity argument akin to 1(c).

(d) If A is invertible, then A is a product of elementary matrices, which all have nonzero determinant, so $\det(A) \in F^{\times}$ by (b). If A is not invertible, then $\det(A) = 0$ by 1(c).

(e) The case where $\det(A) = 0$ or $\det(B) = 0$ follows from (c) and (d). The case where $\det(A) \neq 0$ and $\det(B) \neq 0$ follows from (d) and (b).

(6) Prove that $\det(A) = \det(A^T)$.

(7) Prove the Laplace expansion formula (along the first column): for $A \in \mathrm{Mat}_{n\times n}(R)$,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \det(\widehat{A}_{i,1}),$$

where $\widehat{A}_{i,1}$ is the $(n-1) \times (n-1)$ matrix obtained from A by removing the ith row and first column.