Determinants

DEFINITION: Let R be a commutative ring, and $A \in \mathrm{Mat}_{n\times n}(R)$. The **determinant** of A is

$$\det(A) = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)}.$$

THEOREM 1: Identify $\mathrm{Mat}_{n\times n}(R)$ with $\underbrace{R^n \times \cdots \times R^n}_{n\text{ times}}$ by considering a matrix as an n-tuple of columns. The determinant is the unique function

$$\det\colon \underbrace{R^n \times \cdots \times R^n}_{n\text{times}} \to R$$

that satisfies the following three properties:

- det is **multilinear**, meaning
$$\det(v_1,\dots,v_{i-1},\boldsymbol{v}+\boldsymbol{w},v_{i+1},\dots,v_n) = \det(v_1,\dots,v_{i-1},\boldsymbol{v},v_{i+1},\dots,v_n) + \det(v_1,\dots,v_{i-1},\boldsymbol{w},v_{i+1},\dots,v_n)$$
$$\det(v_1,\dots,v_{i-1},\boldsymbol{rv},v_{i+1},\dots,v_n) = \boldsymbol{r}\det(v_1,\dots,v_{i-1},\boldsymbol{v},v_{i+1},\dots,v_n)$$
- det is **alternating**, meaning
$$\det(v_1,\dots,v_n) = 0 \qquad \text{if } v_i = v_j \text{ for some } i \neq j.$$
- $\det(e_1,\dots,e_n) = 1$.

(1) Working with Theorem 1:
- **(a)** Use Theorem 1 to explain why the determinant of a diagonal matrix is the product of its diagonal entries.
- **(b)** Use Theorem 1 to show that if some column of A is a linear combination of the other columns of A, then $\det(A) = 0$.
- **(c)** Use part (b) to show that if $R = F$ is a field, and A is not invertible, then $\det(A) = 0$.
- **(d)** Use Theorem 1 to show[1] that
$$\det(\boldsymbol{v_2},\boldsymbol{v_1},v_3,\dots,v_n) = -\det(\boldsymbol{v_1},\boldsymbol{v_2},v_3,\dots,v_n).$$
Likewise, the same holds for swapping any two entries.

(2) Uniqueness part of Theorem 1:
- (a) Use 1(d) to show that for any $\sigma \in S_n$,
$$\det(v_{\boldsymbol{\sigma(1)}},\dots,v_{\boldsymbol{\sigma(n)}}) = \mathrm{sgn}(\sigma)\det(v_1,\dots,v_n).$$
- (b) Explain the following claim: if $F\colon \underbrace{R^n \times \cdots \times R^n}_{n\text{times}} \to R$ is multilinear, then F is completely determined by $F(e_{i_1},\dots,e_{i_n})$ for $1 \le i_1,\dots,i_n \le n$.
- (c) Explain the following claim: if $F\colon \underbrace{R^n \times \cdots \times R^n}_{n\text{times}} \to R$ is multilinear and alternating, then F is completely determined by $F(e_1,\dots,e_n)$.

[1] Hint: Consider $\det(\boldsymbol{v_1}+\boldsymbol{v_2},\boldsymbol{v_1}+\boldsymbol{v_2},v_3,\dots,v_n)$.

THEOREM 2: Let R be a commutative ring and $A, B \in \mathrm{Mat}_{n\times n}(R)$. Then
$$\det(AB) = \det(A)\det(B).$$

PROPOSITION: Let R be a commutative ring. Let A be a square matrix, and B be a matrix obtained from A by an elementary column operation.

- For the operation "add $r \in R$ times column i to column j" we have $\det(B) = \det(A)$.
- For the operation "multiply column i by $u \in R^\times$" we have $\det(B) = u\det(A)$.
- For the operation "swap column i and column j" we have $\det(B) = -\det(A)$.

(3) Use Theorem 1 to prove the Proposition.

(4) Use the Proposition (and not the definition) to compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 7 & 13 \end{bmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{Q})$.

(5) Proof of Theorem 2 in the case $R = F$ is a field:

- **(a)** Prove Theorem 2 in the case $B = E$ is an elementary matrix.
- **(b)** Prove[2] Theorem 2 in the case A and B are both invertible matrices.
- **(c)** Show that AB is invertible if and only if A and B are both invertible.
- **(d)** Show[3] that $\det(A) \in F^\times$ if and only if A is invertible.
- **(e)** Complete the proof of Theorem 2 in the field case.

(6) Prove that $\det(A) = \det(A^T)$.

(7) Prove the Laplace expansion formula (along the first column): for $A \in \mathrm{Mat}_{n\times n}(R)$,
$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \det(\widehat{A}_{i,1}),$$
where $\widehat{A}_{i,1}$ is the $(n-1)\times(n-1)$ matrix obtained from A by removing the ith row and first column.

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[2]Hint: You can use the fact that over a field, every invertible matrix is a product of elementary matrices

[3]Hint: Use part (a) and 1(c).