DEFINITION: Let R be a commutative ring. Let V be a free R-module with ordered basis $\mathcal{B} = \{b_1, \dots, b_n\}$ and let W be a free R-module with ordered basis $\mathcal{C} = \{c_1, \dots, c_m\}$. Given an R-module homomorphism $T \colon V \to W$, for each $j = 1, \dots n$, write

(♣) $$T(b_j) = r_{1,j}c_1 + \cdots + r_{m,j}c_m$$

for some elements $r_{i,j} \in R$. The matrix

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m,1} & r_{m,2} & \cdots & r_{m,n} \end{bmatrix}$$

is the **matrix representing T in the bases $\mathcal{B}$ and $\mathcal{C}$.**

(1) Warming up with the definition:
- **(a)** If R is a field F, translate everything[1] in the definition into linear algebra terms.
- **(b)** Use the equation (♣) to explain as concretely as possible what the j-th column of $[T]_{\mathcal{B}}^{\mathcal{C}}$ means in terms of T, $\mathcal{B}$, and $\mathcal{C}$.
- **(c)** Explain why the entries $r_{i,j}$ are well-defined.
- **(d)** *Just using your answer for part **(b)** and not looking at the formula*, describe the dimensions of the matrix $[T]_{\mathcal{B}}^{\mathcal{C}}$ in terms of the rank of V and the rank of W.
- **(e)** Let V be the $\mathbb{R}$-vector space of polynomials in $\mathbb{R}[x]$ of degree at most 3 along with the zero polynomial. The derivative map $\frac{d}{dx}$ is a linear transformation from V to V. Choose a nice basis $\mathcal{B}$ for V and compute the matrix $[\frac{d}{dx}]_{\mathcal{B}}^{\mathcal{B}}$.
- **(f)** Find another[2] basis $\mathcal{C}$ for V such that $[\frac{d}{dx}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

- **(a)** Let F be a field. Let V be an F-vector space with ordered basis $\mathcal{B} = \{b_1, \dots, b_n\}$ and let W be a free R-module with ordered basis $\mathcal{C} = \{c_1, \dots, c_m\}$. Given an F-linear transformation $T \colon V \to W$ (the rest is the same).
- **(b)** The j-th column of $[T]_{\mathcal{B}}^{\mathcal{C}}$ is the expression of the image of the jth basis vector in $\mathcal{B}$ as a linear combination of $\mathcal{C}$.
- **(c)** This is the uniqueness of expression in terms of a basis applied to $\mathcal{C}$.
- **(d)** The number of columns equals the rank of V, since there is one column for each basis vector in $\mathcal{B}$. The number of rows is the number of entries in a column which is the rank of W, since in each column we have one coefficient for each element of $\mathcal{C}$.
- **(e)** One possibility is $\mathcal{B} = \{x^3, x^2, x, 1\}$, and the matrix is
$$[\frac{d}{dx}]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$
- **(f)** Take $\mathcal{C} = \{3x^2, 2x, 1, x^3\}$.

(2) Show that if $\mathcal{E} = \{e_1, \dots, e_n\}$ is the standard basis on R^n and $\mathcal{E}' = \{e_1, \dots, e_m\}$ is the standard basis on R^m, then $T(v) = [T]_{\mathcal{E}}^{\mathcal{E}'} \cdot v$, where the RHS is usual matrix-times-vector multiplication.

[1] You can do this aloud instead of rewriting everything.

[2] You might have to reorder or change $\mathcal{B}$ if you are unlucky.

PROPOSITION: Let R be a commutative ring. Let V be a free R-module with ordered basis $\mathcal{B} = \{b_1, \dots, b_n\}$ and let W be a free R-module with ordered basis $\mathcal{C} = \{c_1, \dots, c_m\}$. Then the map

$$\begin{array}{ccc} \mathrm{Hom}_R(V, W) & \longrightarrow & \mathrm{Mat}_{m\times n}(R) \\ T & \longmapsto & [T]_{\mathcal{B}}^{\mathcal{C}} \end{array}$$

is bijective. Moreover, this is an isomorphism of R-modules.
When $V = W$ and $\mathcal{B} = \mathcal{C}$, the same map

$$\begin{array}{ccc} \mathrm{End}_R(V) & \longrightarrow & \mathrm{Mat}_{n\times n}(R) \\ T & \longmapsto & [T]_{\mathcal{B}}^{\mathcal{B}} \end{array}$$

is an isomorphism of rings.

(3) Prove that the map $T \mapsto [T]_{\mathcal{B}}^{\mathcal{C}}$ in the Proposition is bijective.

To see that this is surjective, we use the UMP for free modules: Given a matrix $A = [a_{i,j}]$, there is an R-module homomorphism $\phi : V \to W$ such that $\phi(b_j) = \sum_i a_{i,j} b_i$, and by definition, $[\phi]_{\mathcal{B}}^{\mathcal{C}} = A$.

To see that this is injective, we use the UMP for free modules: Suppose that $[\phi]_{\mathcal{B}}^{\mathcal{C}} = [\psi]_{\mathcal{B}}^{\mathcal{C}} = A$. Then $\phi(b_j) = \sum_i a_{i,j} b_i = \psi(b_j)$ for all j. Then the uniqueness part of the UMP says that $\phi = \psi$. This shows that the map is injective.

(4) Suppose that V is a free module with a countably infinite basis $\mathcal{B} = \{b_1, b_2, b_3, \dots\}$, and W is free with a countably infinite basis $\mathcal{C} = \{c_1, c_2, c_3, \dots\}$. What is the analogue of the Proposition in this case?

(5) Prove the Proposition.