PROPOSITION: Let R be a ring and F be a free module with basis B. Then every element of $f \in F$ admits a unique expression as a linear combination[1] of elements of B.

UNIVERSAL MAPPING PROPERTY FOR FREE MODULES: Let R be a ring and F be a free module with basis B. Let N be an arbitrary R-module. Then for any function $j : B \to N$, there is a unique R-module homomorphism $h : F \to N$ such that $h(b) = j(b)$ for all $b \in B$.

(1) Let R be a ring and $n \in \mathbb{Z}_{>0}$. The **standard free module of rank n** and its **standard basis** are, respectively,

$$R^n = \left\{ \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \middle| \, r_i \in R \right\} \quad \text{and} \quad \text{the set with elements } e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

We also write elements in the form $(r_1, \dots, r_n)$.

(a) Let $R = \mathbb{Z}[x]$ and $M = R^3$. Give the unique expression of $v = (2x + 3, 1, x^4)$ as a linear combination of the standard basis.

(b) Let $R = \mathbb{Z}[x]$, and $M = R^3$, and $N = \mathbb{Z}/5[x]$. Let $h : M \to N$ be the unique R-linear map such that $h(e_1) = [2]$, $h(e_2) = [0]$, and $h(e_3) = x$. Compute $h(v)$.

(a) $v = (2x + 3)e_1 + e_2 + x^4 e_3$.

(b) $h(v) = (2x + 3)[2] + [0] + x^4 \cdot x = x^5 + [4]x + [1]$.

(2) Proving things.

(a) Prove the Proposition above.

(b) Prove the UMP for free modules above.

(a) Let $f \in F$. Since B is a basis, there is at least one expression $f = r_1 b_1 + \cdots + r_n b_n$ with $r_i \in R$ and $b_i \in B$ because B generates F. Given another, by including some extra zero coefficients (to both expressions), we can assume the other uses the same elements of B, so take $f = r'_1 b_1 + \cdots + r'_n b_n$. Then after subtracting we get

$$0 = f - f = \sum_i r_i b_i - \sum_i r'_i b_i = \sum_i (r_i - r'_i) b_i,$$

so $r_i - r'_i = 0$ by linear independence, and hence $r_i = r'_i$ for all i. This shows uniqueness.

(b) First we show uniqueness. By the Proposition, we can write $f = \sum_i r_i b_i$ in a unique way, and we must have $h(f) = h(\sum_i r_i b_i) = \sum_i r_i h(b_i) = \sum_i r_i j(b_i)$. This gives a unique value for each $f \in F$. For existence, we check that the function given by this formula is an R-module homomorphism. To do it, let $f, f' \in F$. Then, after adding some zero coefficients if necessary, we can write $f = \sum_i r_i b_i$ and $f' = \sum_i r'_i b_i$. Then $f + f' = \sum_i (r_i + r'_i) b_i$, suing the module axioms. We then have

$$h(f + f') = \sum_i (r_i + r'_i) j(b_i) = \sum_i r_i j(b_i) + \sum_i r'_i j(b_i) = h(f) + h(f').$$

The check that h is compatible with multiplication by scalars is similar.

[1] Recall that a linear combination of B is a sum of the form $r_1 b_1 + \cdots + r_n b_n$ for some finite list of elements $b_1, \dots, b_n \in B$ and $r_1, \dots, r_n \in R$.

THEOREM: Let R be a ring. Let F be a free module with a basis B, and F' be a free module with a basis B'.

(1) If $|B| = |B'|$, meaning there is a set bijection between B and B', then $F \cong F'$.
(2) Let R be a commutative ring. If $F \cong F'$, then $|B| = |B'|$.

DEFINITION: Let R be a commutative ring, and F be a free module. The **rank** of F is the size of a basis B of F.

(3) Rank:

- **(a)** What about the Definition above needs justification? Use the Theorem to justify it.
- **(b)** Prove part (1) of the Theorem. (We will prove part (2) later as a consequence of the same result in the special case of vector spaces.)

- **(a)** That every basis has the same size. That follows from part (2) in the case $F = F'$.
- **(b)** Let $j : B \to B'$ be a bijection. Since $B' \subseteq F'$, by the UMP for free modules, there is a unique R-linear map $h : F \to F'$ such that $h|_B = j$. Then, using the inverse map $j^{-1} : B' \to B$, since $B \subseteq F$, the UMP for free modules gives us a unique R-linear map $h' : F' \to F$ such that $h'|_{B'} = j^{-1}$. Consider the composition $h' \circ h : F \to F$. Its restriction to B is the identity map. Thus, again by UMP, $h' \circ h$ is the identity on F. Along similar lines, the composition $h \circ h' : F' \to F'$ is the identity. It follows that $F \cong F'$.

(4) Let $A = M_\infty(\mathbb{R})$ be the ring of countably infinite matrices with real entries:

$$M_\infty(\mathbb{R}) = \{[a_{ij}]_{\substack{i=1,2,3,\dots \\ j=1,2,3,\dots}} \mid a_{ij} \neq 0 \text{ for at most finitely many pairs } (i,j)\}$$

with usual matrix addition and multiplication; you do not have to prove that this is a ring. Prove[2] that $A^1 \cong A^2$ as A-modules. What does this say about the Theorem?

[2]Hint: Consider the map sending a matrix $[a_{ij}]$ to the pair of matrices $([a_{i,2j-1}], [a_{i,2j}])$ reconstituted from its odd columns and its even columns.