Some old qualifying exam questions

Problem 1. Consider $f(x) = x^6 + 3 \in \mathbb{Q}[x]$.

(a) Let[1] a be a root of $f(x)$ and prove $\mathbb{Q}(a)$ is a Galois field extension of $\mathbb{Q}$.

(b) Find the Galois group $\mathrm{Gal}(\mathbb{Q}(a)/\mathbb{Q})$.

Problem 2. Let A be the matrix with entries in $\mathbb{C}$

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

(a) Find the Jordan canonical form of A.

(b) Is A similar to

$$\begin{bmatrix} 0 & -9 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 1 & 6 \end{bmatrix}?$$

Problem 3. Suppose $F \subseteq K$ is a finite extension of fields such that $[K : F]$ is odd. Show that for any $b \in K$ we have $F(b) = F(b^2)$.

Problem 4. Find, with justification, a complete and non-redundant list of conjugacy class representatives for the group $\mathrm{GL}_3(\mathbb{F}_2)$, where $\mathbb{F}_2$ is the field with two elements.

Problem 5. Let p be any positive prime integer.

(a) Prove that if $p = k^2 + 1$ for some integer k, then p is not an irreducible element of $\mathbb{Z}[i]$.

(b) Prove $x^4 - p$ is irreducible in $\mathbb{Q}(i)[x]$. *Tip:* Find the degree of the splitting field of $x^4 - p$ over $\mathbb{Q}(i)$.

Problem 6. Let E/F be a field extension and $\sigma : E \to E$ a field homomorphism fixing F; i.e., $\sigma(f) = f$ for all $f \in F$.

(a) Prove that if E is algebraic over F then σ is an isomorphism.

(b) Give an example of a specific field extension E/F and a homomorphism $\sigma : E \to E$ fixing F that is not an isomorphism.

Problem 7. Let L/F be a finite Galois extension with Galois group G. Show that $L = F(\alpha)$ if and only if the images of α under G are distinct.

[1] Hint: First show $\frac{a^3+1}{2}$ is primitive 6-th root of unity.