Problem Set 2

Due Thursday, January 29

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

Problem 1. Let R be a ring, let M be an R-module, and N be a submodule of M.

(a) Suppose that M is free with basis B. Suppose that N is free with basis $C \subseteq B$. Show that M/N is free and give a formula for a basis in terms of B and C.

(b) Suppose that N is free and M/N is free. Show that M is free.

Problem 2. Let R be a ring, let M be an R-module, and N be a submodule of M.

(a) Give an example of a free module M and a submodule N that is free but M/N is not free.

(b) Give an example of a free module M and a submodule N that is not free.

Problem 3. A module M is **simple** if the only submodules of M are 0 and M. Let R be a commutative ring. Show that an R-module M is simple if and only if $M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.

Problem 4. Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.