Problem Set 13

Due never

Problem 1. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is a Galois extension of degree 4 with Galois group that is a cyclic group of order 4.

Problem 2. Let $F \subseteq L$ be Galois field extension of degree 45. Prove there exists a unique intermediate field E such that $[E : F] = 5$.

Problem 3. Let $f(x) = x^6 - 7 \in \mathbb{Q}[x]$ and let E be a splitting field of $f(x)$ over $\mathbb{Q}$. Let $\omega = e^{2\pi i/6}$ be a primitive 6th root of unity.

(a) Find, with justification, $[E : \mathbb{Q}]$.

(b) Prove that $f(x)$ is irreducible over $\mathbb{Q}(\omega)$.

(c) Prove that $\mathrm{Aut}(E/\mathbb{Q})$ has a normal subgroup of order 3.

Problem 4. Let F be a field with p^6 elements for some prime p. Determine the number of subfields of F, and compute the number of elements in each such subfield.

Problem 5. Let F be a field of characteristic $p > 0$, $a \in F$, and consider the polynomial $f(x) = x^p - x - a \in F[x]$.

(a) Prove[1] that $f(x)$ is either irreducible over F or it splits into distinct linear factors over F.

(b) Suppose $f(x)$ is irreducible over F and let L be a splitting field of f over F. Prove that the Galois group of L over F is cyclic.

[1] Hint: If α is a root of $f(x)$, consider $\alpha + j$ for $j \in \mathbb{Z}/p$.