PROBLEM SET #3

(1) Is $\mathbb{Z}[\sqrt[3]{37}]$ a regular ring? What about $\mathbb{Z}[\sqrt[3]{43}]$?

(2) Let R be an A-algebra, $f(x_1, \dots, x_n) \in A[x_1, \dots, x_n]$ a polynomial with coefficients in A, and $r_1, \dots, r_n, s_1, \dots, s_n \in R$.
  (a) Prove the *chain rule* for the universal derivation: $d_{R|A}(f(r_1, \dots, r_n)) = \sum_i \frac{df}{dx_i}(r_1, \dots, r_n) dr_i$.
  (b) Prove the *Taylor expansion* formula: $f(r_1+s_1, \dots, r_n+s_n) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{|\alpha|!} \frac{d^{|\alpha|} f}{dx_1^{\alpha_1} \cdots dx_n^{\alpha_n}}(r_1, \dots, r_n) s_1^{\alpha_1} \cdots s_n^{\alpha_n}$.

(3) Facts about p-bases/ p-degree:
  (a) Let L be a field of positive characteristic. Let T be a p-basis for L. Show that for any e, the set $T^{[<p^e]}$ is a basis for L.
  (b) Let $K \subseteq L$ be a finite extension of fields of positive characteristic. Show that $p\deg(K) = p\deg(L)$.
  (c) Let $L = K(x_1, \dots, x_m)$ be a field of rational functions in m variables over K. Show that $p\deg(L) = p\deg(K) + m$.

(4) Let k be a field of positive characteristic with a finite p-basis, R be a finitely generated k-algebra, and $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals of R. Show that

$$\dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} = p\deg(\kappa(\mathfrak{p})) - p\deg(\kappa(\mathfrak{q})).$$

(5) Let K be a field.
  (a) Let $R = K[x]$ be a polynomial ring in one variable and $M = R^{\oplus \mathbb{N}}$ be a free R-module on a countable basis. Compute the (x)-adic completion of M.
  (b) Let $R = K[x_1, x_2, \dots]$ be a polynomial ring in countably many variables and $\mathfrak{m} = (x_1, x_2, \dots)$. Describe the elements of $\hat{R}^{\mathfrak{m}}$. Find an element in the maximal ideal of $\hat{R}^{\mathfrak{m}}$ that is *not* an element of $\mathfrak{m}\hat{R}^{\mathfrak{m}}$.

(6) Let $K \subseteq L$ be an extension of fields.
  (a) Suppose that L is a finitely generated over K as fields. Show that L is formally unramified over K if and only if the extension is separable algebraic.
  (b) Show that the finite generation hypothesis is strictly necessary in part (1).