**PROBLEM SET #2**

(1) Compute the singular locus (i.e., set of points at which the variety is singular) of each of the following complex affine varieties[1]:

(a) $V(x^2 - y^2 z) \subseteq \mathbb{C}^3$.

(b) $V(x^2 + y^2 + z^2 - 1) \subseteq \mathbb{C}^3$.

(c) $V\left(2 \times 2 \text{ minors of } \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}\right) \subseteq \mathbb{C}^6$.

(2) Determine if each of the following local rings is regular or not:

(a) $R = \mathbb{Z}[x, y]_{(2,x,y)}/(2 - x^2 + y^2)$.

(b) $R = \mathbb{Z}[x, y]_{(2,x,y)}/(4 - x^2 + y^2)$.

(3) * Let $X = V(f_1, \dots, f_a) \subseteq \mathbb{C}^m$ and $Y = V(g_1, \dots, g_b) \subseteq \mathbb{C}^n$ be complex affine varieties. Let $\Phi : X \to Y$ be the morphism given by the rule $\Phi(a_1, \dots, a_m) = (h_1(\underline{a}), \dots, h_n(\underline{a}))$ for some polynomials $h_1, \dots, h_n \in \mathbb{C}[x_1, \dots, x_m]$. Let $S = \mathbb{C}[X]$ and $R = \mathbb{C}[Y]$ be corresponding coordinate rings, $\mathfrak{n}$ the maximal ideal of $a$, $\mathfrak{m} = \mathfrak{n} \cap R$ the maximal ideal of $\Phi(a)$, and $\phi : R \to S$ the induced map (i.e., $\phi(y_i) = h_i(\underline{x})$).

(a) Show that the map $T_a(X) \xrightarrow{\xi_{X,a}} \mathrm{Der}_{R|\mathbb{C}}(R/\mathfrak{n})$ is an isomorphism.

$$v \longmapsto \sum_i v_i \frac{d}{dx_i}|_{x=a}$$

(b) Show that there is a well-defined vector space map $T_a(\Phi) : T_a(X) \to T_{\Phi(a)}(Y)$ given by $T_a(\Phi)(v) = J(h_1, \dots, h_n)|_a \cdot v$.

(c) Note that $R/\mathfrak{m} \cong S/\mathfrak{n}$, so we can identify $S/\mathfrak{n}$ as an $R$-module with $R/\mathfrak{m}$. Show that the following diagram commutes

$$\begin{array}{ccc} T_a(X) & \xrightarrow{T_a(\Phi)} & T_{\Phi(a)}(Y) \\ \cong \downarrow \xi_{X,a} & & \cong \downarrow \xi_{Y,\Phi(a)} \\ \mathrm{Der}_{S|\mathbb{C}}(S/\mathfrak{n}) & \xrightarrow{\phi^*} & \mathrm{Der}_{R|\mathbb{C}}(R/\mathfrak{m}). \end{array}$$

(4) * Let $K$ be an algebraically closed field. Let $R \xrightarrow{\phi} S$ be a homomorphism of finitely generated reduced $K$-algebras, and $X \xrightarrow{\Phi} Y$ the corresponding map of varieties. Show that $\Omega_{S|R} = 0$ if and only if the induced map on tangent spaces $T_a(\Phi) : T_a(X) \to T_{\Phi(a)}(Y)$ is injective for every $a \in X$.

(5) Let $R$ and $S$ be $A$-algebras.

(a) Show that $\Omega_{(R \times S)|A} \cong \Omega_{R|A} \oplus \Omega_{S|A}$.

(b) Show that $\Omega_{(S \otimes_A R)|S} \cong S \otimes_A \Omega_{R|A}$.

(6) (a) * Show that if $R$ has characteristic $p > 0$, then $\Omega_{R|\mathbb{Z}} = \Omega_{R|R^p}$.

(b) Show that $\Omega_{\mathbb{F}_p[\![x]\!]|\mathbb{F}_p}$ is generated by $dx$.

(c) Is $\Omega_{\mathbb{C}[\![x]\!]|\mathbb{C}}$ generated by $dx$? Is it finite generated?

[1]You do not need to justify that these are irreducible, and that the dimension of the third is four; however, you should know how to do this!