

MIDTERM EXAM

Please turn in *four* of the following problems. If you intend to take an written algebra comprehensive exam, I recommend attempting the problems in a timed setting with no notes at first, and then continuing with the problems later.

- (1) For each of the following, find an example with justification, or show that none exists:
 - (a) A ring R such that $\mathbb{C} \subseteq R \subseteq \mathbb{C}[x, y]$ that is not Noetherian.
 - (b) A ring R such that $\mathbb{C}[x^2, y^2] \subseteq R \subseteq \mathbb{C}[x, y]$ that is not Noetherian.
- (2) Let R be a unique factorization domain and $\mathfrak{p} \subseteq R$ be a nonzero *prime* ideal. Show that \mathfrak{p} has height 1 if and only if it is principal.
- (3) Let (R, \mathfrak{m}) be a local ring and M be a nonzero R -module. Assume that R is a domain *but not a field*. Show that if M is injective then M is not finitely generated.
- (4) Give, with justification, ideals $I, J \subseteq \mathbb{C}[x, y]$ such that
 - $\sqrt{I} = \sqrt{J} = (y)$,
 - I has a unique minimal primary decomposition,
 - J has distinct minimal primary decompositions.
- (5) Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of modules. Show that $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$.
- (6) Let M be a finitely generated module over a Noetherian ring R . Show¹ that the following are equivalent:
 - (a) M has finite length;
 - (b) $\text{Ass}_R(M) \subseteq \text{Max}(R)$;
 - (c) $\text{Supp}_R(M) \subseteq \text{Max}(R)$.
- (7) Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be ring homomorphisms. Show that $A \xrightarrow{\beta\alpha} C$ is integral if and only if $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ are both integral.

¹Hint: For (3) \Rightarrow (1), you may want to apply the previous problem to a prime filtration.