ASSIGNMENT #5

(1) A *magic square* of size n and row sum t is an $n \times n$ matrix with entries in $\mathbb{Z}_{\geqslant 0}$ such that each row and each column adds up to t. In this problem, we will prove that the for a fixed n, the number of $n \times n$ magic squares with row sum t is eventually equal to a polynomial of degree ?

Let K be a field, $X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$ be a matrix of indeterminates, and $R = K[X]$ be a polynomial ring. For any $n \times n$ matrix A with entries in $\mathbb{Z}_{\geqslant 0}$, write X^A for the monomial $x_{11}^{A_{11}} \cdots x_{nn}^{A_{nn}}$.

A *permutation matrix* is a matrix such that for each row and each column, there is only one nonzero entry, and the value of that entry is one. Equivalently, a permutation matrix is a magic square with row sum one.

(a) Show that every magic square is a sum of permutation matrices.

(b) Show that the K-vector subspace S of R with basis

$$\{X^A \mid A \text{ is a magic square}\}$$

is a K-subalgebra of R, and that S generated as a K-algebra by the set

$$\{X^A \mid A \text{ is a permutation matrix}\}.$$

(c) Show that S is a standard graded K-algebra if one sets

$$\deg(X^A) = (a_{11} + a_{12} + \cdots + a_{1n} + \cdots + a_{nn})/n.$$

(d) Compute the dimension of S.

(e) Explain how the conclusion follows.

(2) Let $\psi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat homomorphism.

(a) Show that[1] for any nonzero R-module M, $S \otimes_R M \neq 0$.

(b) Show that[2] ψ is injective.

(c) Show that[3] the conclusion of the "Going Down" theorem holds for ψ.

(d) Show that $\dim(R) + \dim(S/\mathfrak{m}S) = \dim(S)$.

[1]Hint: NAK

[2]Hint: Show that for any ideal $J \subset R$, $J \otimes_R S \cong JS$, and let $J = \ker(\psi)$.

[3]Hint: Show that $R_\mathfrak{p}/\mathfrak{p}'R_\mathfrak{p} \to S_\mathfrak{q}/\mathfrak{p}'S_\mathfrak{q}$ is flat. It might help to note that if M is a flat A-module and B is a flat A-algebra, then $B \otimes_A M$ is a flat B-module.