ASSIGNMENT #4

(1) Let K be a field, and R be a positively graded K-algebra. Let M be[1] an $\mathbb{N}$-graded R-module.
   (a) Show that $(R_+)M = M$ implies $M = 0$.
   (b) Show that for a subset *of homogeneous elements* $S \subset M$, M is generated by S as an R-module if and only if $M/(R_+)M$ is generated by the images of the elements of S as a K-vector space.

(2) Compute the dimension of each of the following rings R, where K is a field and x, y, z, u, v are indeterminates:
   (a) $R = \dfrac{K[x, y, z]}{(x^3, x^2y, xyz)}$.
   (b) $R = K[x^2u, xyu, y^2u, x^2v, xyv, y^2v] \subseteq K[x, y, u, v]$.
   (c) $R = \dfrac{K[x, y, z, u, v]}{(x^3u^2 + y^3uv + z^3v^2)}$.
   (d) $R = \dfrac{K[x, y, u, v]}{(u^3 - xy, v^5 - x^2u - y^3)}$.

(3) Let K be a field, and $R \subseteq S$ be a module-finite inclusion of finitely generated K-algebras that are both domains[2]. Show that for any $\mathfrak{q} \in \operatorname{Spec}(S)$, $\operatorname{height}(\mathfrak{q}) = \operatorname{height}(\mathfrak{q} \cap R)$.

(4) Let $\psi : R \hookrightarrow S$ be an algebra-finite inclusion of rings.
   (a) Show that if R is a domain, then $\operatorname{im}(\psi^*)$ contains a nonempty open subset of $\operatorname{Spec}(R)$.
   (b) Show that[3] for every minimal prime $\mathfrak{p}$ of R, $\operatorname{im}(\psi^*)$ contains a nonempty open subset of $V(\mathfrak{p})$.

(5) Let $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$ be a 2×3 matrix of indeterminates over $\mathbb{C}$ and $R = \mathbb{C}[X]$. Let I be the ideal of 2×2 minors of X. Show[4] that $\mathbb{C}[x_{11}, x_{12} - x_{21}, x_{13} - x_{22}, x_{23}]$ is a Noether normalization for R/I, and conclude that the height of I is two.

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[1] Note that we are not assuming that M is finitely generated.
[2] Note that we are not assuming that R is normal.
[3] First show that each minimal prime $\mathfrak{p}$ is in the image of $\operatorname{im}(\psi^*)$, so $\mathfrak{p}S \cap R = \mathfrak{p}$. To see this, you may want to consider the localization of the map ψ at $(R \smallsetminus \mathfrak{p})$.
[4] Hint: You may want to use the problem (1) to show that that the map is module-finite.