ASSIGNMENT #3

(1) Let R be a Noetherian ring and W^{-1} be a multiplicatively closed subset. Show that $W^{-1}R$ is also Noetherian.

(2) Let $K \subseteq L$ be fields, and $R = K[x_1, \ldots, x_n] \subseteq S = L[x_1, \ldots, x_n$ be polynomial rings. Let $I \subseteq R$ be an ideal.

(a) Show[1] that if $I \subseteq R$ is a proper ideal, then the expansion $IS \subseteq S$ is also a proper ideal.

(b) Show[2] that $IS \cap R = I$.

(3) Let K be a field, $R = K[x_1, \ldots, x_n]$ be a polynomial ring, and $I \subseteq R$ be a proper ideal.

(a) Show that $\mathcal{Z}_{\overline{K}}(I) \neq \varnothing$.

(b) Show that $f \in \sqrt{I}$ if and only if $\mathcal{Z}_{\overline{K}}(f) \supseteq \mathcal{Z}_{\overline{K}}(I)$.

(c) Show[3] that

$$\sqrt{I} = \bigcap_{\mathfrak{m} \in \operatorname{Max}(R) \cap V(I)} \mathfrak{m}.$$

(4) Describe all of the elements of $\operatorname{Spec}(\mathbb{C}[x])$ and $\operatorname{Spec}(\mathbb{R}[x])$, and describe the map $\operatorname{Spec}(\mathbb{C}[x]) \to \operatorname{Spec}(\mathbb{R}[x])$ induced by the inclusion map $\mathbb{R}[x] \subseteq \mathbb{C}[x]$.

(5) Let R be Noetherian, and M be an R-module, not necessarily finitely generated. Show that the support of M is Zariski closed implies that $\operatorname{Ass}_R(M)$ has finitely many minimal elements.

(B) Show[4] that $\operatorname{Spec}(R)$ is connected if and only if R cannot be written (up to isomorphism) as a direct product of two rings.

[1] Hint: It might be useful to think of $L[x_1, \ldots, x_n]$ as $L \otimes_K K[x_1, \ldots, x_n]$.

[2] Hint: Consider the map $R/I \to S/IS$.

[3] Hint: For the containment "$\supseteq$", if $f \notin \sqrt{I}$, take $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{Z}_{\overline{K}}(I) \smallsetminus \mathcal{Z}_{\overline{K}}(f)$. Show that the image of the ring homomorphism $\phi_{\mathbf{a}} : R \to \overline{K}$ sending $f \mapsto f(a_1, \ldots, a_n)$ is a field, and hence the kernel is a maximal ideal containing I but not f.

[4] Hint: You might end up wanting to show along the way that if the class of e in $R/\sqrt{(0)}$ is idempotent, then some element in $e + \sqrt{(0)}$ is idempotent in R. It may be helpful to apply the Chinese Remainder Theorem to $e^n(1-e)^n$ for this.