ASSIGNMENT #1

(1) Let R be a ring, and M be an R-module. The *Nagata idealization* of M is the ring[1] $R \ltimes M$ that
- as an (additive) abelian group, is given by $R \ltimes M := R \oplus M$, and
- the multiplication is $(r, m) \cdot (s, n) := (rs, rn + sm)$.

Let I be an ideal of a ring R that is not finitely generated. Show that $R \subseteq R \ltimes I \subseteq R \ltimes R$, that $R \subseteq R \ltimes R$ is module-finite, but $R \subseteq R \ltimes I$ is not module-finite.

(2) Let K be a field, x be an indeterminate, and R be a ring such that $K \subseteq R \subseteq K[x]$. Show that R is a finitely generated K-algebra. What if K is not a field?

(3) Find a finite generating set for $\mathbb{R}[x, y, z]$ as a $\mathbb{R}[x^2 + yz, x + y, z]$-module.

(4) Noetherian rings and prime ideals:
(a) Show that if every prime ideal of R is finitely generated, then R is Noetherian.[2]
(b) If the set of prime ideals of R satisfies ACC, must R be Noetherian?

(5) Show that if R is a unique factorization domain, then R is integrally closed in its fraction field.

(B) Let $A \subseteq B$ be a module-finite inclusion of rings. If B is Noetherian, then so is A. (Compare this to the Artin-Tate Lemma, which says that if B is algebra-finite over some subring $A_0 \subseteq A$, then so is A.)[3]

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[1] You can assume that this satisfies the ring axioms, but you may wish to check a couple for yourself.
[2] Hint: If not, take an ideal I that is maximal among the ideals that are not finitely generated, take $f, g \notin I$ with $fg \in I$, and consider the ideals (I, f) and $I : f := \{r \in R \mid rf \in I\}$.
[3] Hint: By a problem above, it suffices to show that B is a Noetherian A-module. Take a maximal element of

$$\{IB \mid I \text{ is an ideal of } A \text{ and } B/IB \text{ is not a Noetherian } A\text{-module}\}$$

and reduce to the case B/aB is a Noetherian A-module for any nonzero $a \in A$. Then take a maximal element of

$$\{N \subseteq B \mid N \text{ is an } A\text{-submodule of } B \text{ and } B/N \text{ has annihilator zero}\}$$

and reduce to the case that for every nonzero A-submodule $M \subseteq B$, there is some $a \in A$ with $a(B/M) = 0$. Then use the short exact sequence

$$0 \to aB \to M \to M/aB \to 0 \dots$$

to argue that M is finitely generated.