FINAL EXAM

Please turn in *four* of the following problems. If you intend to take a written algebra comprehensive exam, I recommend attempting the problems in a timed setting with no notes at first, and then continuing with the problems later.

(1) Let K be a field and $S = K[x_1, \ldots, x_n]$ be a polynomial ring over K. Let R be any subring of S that contains every polynomial $f \in S$ with the property that

$$f(x_1, x_2, \ldots, x_{n-1}, x_n) = f(x_2, x_3, \ldots, x_n, x_1).$$

Show that R is a finitely generated K-algebra.

(2) Let $K \subseteq L$ be fields, and R be a finitely generated K-algebra. Determine if each of the following is true or false, and justify with a proof or counterexample.
   (a) If R is a domain, then $L \otimes_K R$ is a domain.
   (b) If $L \otimes_K R$ is a domain, then R is a domain.
   (c) If R and $L \otimes_K R$ are domains, then $\dim(R) = \dim(L \otimes_K R)$.

(3) Let R be a Noetherian ring. Let M be a nonzero R-module (not necessarily finitely generated!), and suppose that $\mathrm{Ass}_R(M)$ has finitely many minimal elements. Show that the support of M is a Zariski closed subset of $\mathrm{Spec}(R)$.

(4) Let R be a domain and F be its fraction field. For an R-module M, the *rank* of M is the dimension of the F-vector space $M_{(0)} \cong F \otimes_R M$. Assume that M is finitely generated.
   (a) Show that the rank of M is finite.
   (b) Show that there is a short exact sequence of the form

$$0 \to R^{\oplus r} \to M \to T \to 0$$

   with $r = \mathrm{rank}(M)$ and T a torsion module.

(5) Let R be a Noetherian ring, and

$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3 = \mathfrak{r}_1 \cap \mathfrak{r}_2 \cap \mathfrak{r}_3$$

be two minimal primary decompositions of an ideal I. Show that[1] after possibly reordering the ideals above, there is a minimal primary decomposition of I of the form

$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{r}_3.$$

[1]Hint: Take $\mathfrak{q}_3, \mathfrak{r}_3$ whose radicals are equal and are not contained in any other element of $\mathrm{Ass}(R/I)$ (and explain why you can). Then use the multiplicative set $W = R \smallsetminus (\sqrt{\mathfrak{q}_1} \cup \sqrt{\mathfrak{q}_2})$.

(6) Let K be a field, and

$$R = K\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}.$$

Let I be the ideal generated by the 2×3 minors of the matrix of variables. Let

$$S = K\begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \end{bmatrix} \subseteq K[u_1, u_2, v_1, v_2, v_3].$$

Consider R as a graded ring with $\deg(x_{ij}) = 1$ and S as a graded ring with $\deg(u_iv_j) = 1$.

(a) Show that there is a surjective graded K-algebra homomorphism $\phi : R/I \to S$ given by sending $\phi(x_{ij}) = u_iv_j$ for all i, j.

(b) Compute the Hilbert function $H_S(t)$ and show[2] that $H_{R/I}(t) \leqslant H_S(t)$ for all t.

(c) Conclude that I is prime.

(7) Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Recall that a local ring is regular if $\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Show that if R is regular then $\mathrm{gr}_{\mathfrak{m}}(R)$ is a polynomial ring.

(Bonus) Show that the conclusion of (3) is false if R is not Noetherian.

[2]Hint: Show that any monomial in R is equivalent modulo I to either

- a monomial in $K[x_{11}, x_{12}, x_{13}, x_{23}]$
- a monomial in $K[x_{11}, x_{12}, x_{22}, x_{23}]$, or
- a monomial in $K[x_{11}, x_{21}, x_{22}, x_{23}]$.