Math 325-001 — Problem Set #9
Due: Monday, April 12 by midnight

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Prove that the function $f(x) = \sqrt{|x|}$ is continuous at every real number a using just the $\epsilon - \delta$ definition of continuity.

(2) Let $f(x)$ be the function with domain all of $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 + x, & \text{if } x \leq 2 \text{ and} \\ 2x - 5, & \text{if } x > 2. \end{cases}$$

Prove f is not continuous at $x = 2$ using just the $\epsilon - \delta$ definition of continuity.

(3) Prove $\sqrt{\sqrt{x^2+1} + x^4 + 1}$ is continuous on all of $\mathbb{R}$. You may use any theorems we've covered in class, but be sure to use them carefully.

(4) Suppose f is a function that is continuous on all of $\mathbb{R}$. Prove[1] that if $f(q) = 0$ for all $q \in \mathbb{Q}$, then $f(x) = 0$ for all x.

(5) Let f be a function defined on $[a, b]$ for some real numbers $a < b$ and assume $f(a) \leq f(b)$. Recall that the Intermediate Value Theorem implies the statement:

If f is continuous on the closed interval $[a, b]$, then for all real numbers y such that $f(a) \leq y \leq f(b)$ there is a real number c such that $a \leq c \leq b$ and $f(c) = y$.

Is the converse of this statement true? If not, give a counterexample; if so, give a proof.

(6) Assume $f(x)$ is continuous on the closed interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$. Prove[2] there is a real number c such that $0 \leq c \leq 1$ and $f(c) = c$. (Such a c is called a "fixed point" of $f(x)$.)

[1] Hint: You may want to use that every real number is the limit of some sequence of rational numbers.

[2] Hint: Apply the Intermediate Value Theorem not to $f(x)$ itself but to a related function. (You may assume that if two functions are both continuous on the closed interval $[0, 1]$, then so is their sum.)