Math 325-001 — Problem Set #8
Due: Monday, April 5 by midnight

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Let $f(x)$ be the function whose domain is all of $\mathbb{R}$ given by the rule

$$f(x) = \begin{cases} 42 & \text{if } x \in \mathbb{Z} \text{ and} \\ 0 & \text{if } x \notin \mathbb{Z}. \end{cases}$$

Prove that for any $a \in \mathbb{R}$, we have $\lim_{x \to a} f(x) = 0$.

(2) Let $f(x)$ be the function whose domain is all of $\mathbb{R}$ given by the rule

$$f(x) = \begin{cases} 42 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \text{ and} \\ 0 & \text{for all other values of } x. \end{cases}$$

Prove $\lim_{x \to 0} f(x)$ does not exist.

(3) Let f be the function whose domain is all of $\mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Prove $\lim_{x \to 0} f(x) = 0$.
(b) Let $a \in \mathbb{R}$ and assume $a \neq 0$. Prove $\lim_{x \to a} f(x)$ does not exist.

Definition: Suppose f is a function and $a \in R$. We say *the limit of $f(x)$ as x approaches a from the right is L* provided:

for all $\epsilon > 0$, there is a $\delta > 0$ such that for all x satisfying $a < x < a + \delta$, we have that f is defined and x and also that $|f(x) - L| < \epsilon$.

In this case we write

$$\lim_{x \to a^+} f(x) = L.$$

(4) Use the definition to prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

(5) Come up with, or look up, the definition of *the limit of $f(x)$ as x approaches a from the left is L*. Use this definition to determine $\lim_{x \to 0^-} f(x)$ where $f(x)$ is the function from problem #2.