## Math 325-001 — Problem Set #7 Due: Monday, March 29 by midnight

**Instructions:** You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

- (1) Suppose  $\{a_n\}_{n=1}^{\infty}$  is a Cauchy sequence such that  $a_{100k} = 0$  for all  $k \in \mathbb{N}$  (i.e., every 100-th term is 0). Prove, using only the definition of "Cauchy", that the sequence converges to 0.
- (2) Consider the sequence  $\{a_n\}_{n=1}^{\infty}$  defined by

$$a_n = 1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \dots + (-1)^{n+1} \frac{1}{4n-3}$$

Equivalently, we may define this sequence recursively by  $a_1 = 1$  and  $a_n = a_{n-1} + (-1)^{n+1} \frac{1}{4n-3}$  for all  $n \ge 2$ .

- (a) Prove that the subsequence of even-indexed terms is strictly increasing and that the subsequence of odd-indexed terms is strictly decreasing; that is, prove  $a_{2k} < a_{2k+2}$  for all  $k \in \mathbb{N}$  and prove  $a_{2k-1} > a_{2k+1}$  for all  $k \in \mathbb{N}$ .
- (b) Prove that every even-indexed term is less than every odd-indexed term; that is, show  $a_{2k} < a_{2j-1}$  for all  $k, j \in \mathbb{N}$ .
- (c) Prove that the sequence is Cauchy and conclude that it therefore converges.
- (d) Bonus: What does it converge to?
- (3) Using just the  $\epsilon \delta$  definition of the limit of a function, prove that

$$\lim_{x \to -3} \frac{x^2 + 5x + 6}{x + 3} = -1.$$

(4) Using just the  $\epsilon - \delta$  definition of the limit of a function, prove that

$$\lim_{x \to 2} x^3 = 8.$$

*Tip*: Use that  $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$  and that whenever |x - 2| < 1, we have  $|x^2 + 2x + 4| < 19$ .

(5) Let a > 0. Using just the  $\epsilon - \delta$  definition of the limit of a function, prove that

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}.$$
  
Tip: Use  $|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}} \le \frac{|x-a|}{\sqrt{a}}.$