Math 325-001 — Problem Set #2
Due: Friday, February 12 by midnight

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Assume S is a subset of $\mathbb{R}$ and that T is a subset of S. Prove that if S is bounded above then T is also bounded above.

(2) Given a subset S of $\mathbb{R}$, a *lower bound* for S is a real number z such that $z \leq s$ for all $s \in S$. We say S is *bounded below* if S has at least one lower bound.

Given a subset S of $\mathbb{R}$, define a new subset $-S$ by

$$-S = \{x \in \mathbb{R} \mid x = -s \text{ for some } s \in S\}.$$

Prove that S is bounded below if and only if $-S$ is bounded above. *Tip*: As with any "if and only if" statement, you need to prove two things: (a) Prove that if S is bounded below, then $-S$ is bounded above, and (b) prove that if $-S$ is bounded above, then S is bounded below.

(3) Suppose S is a subset of $\mathbb{R}$. A real number y is called the *infimum* (also known as *greatest lower bound*) of S if
- y is a lower bound for S
- if z is any lower bound for S then $z \leq y$.

Prove that every nonempty, bounded below subset S of $\mathbb{R}$ has an infimum. *Tip*: Apply the Completeness Axiom to the subset $-S$ defined as in the previous problem.

(4) Prove that if ϵ is any real number such that $\epsilon > 0$, then there exists a natural number n such that $0 < \frac{1}{n} < \epsilon$. *Tip*: You will need to use the following fact, proven in lecture: If x is any real number, then there is a natural number n such that $n > x$.

(5) Let S be the set $\{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$. Prove that 1 is the supremum (aka least upper bound) of S. *Tips*: First show 1 is an upper bound, and then use a proof by contradiction. That is, assume b is an upper bound of S such that $b < 1$ and proceed to derive a contradiction. The statement proven in the previous exercise can be used.

(6) Let S be a subset of $\mathbb{R}$. An element $y \in S$ is called the *minimum element* of S if $y \in S$ and y is a lower bound for S.
   (a) Show that the interval $(0, \infty) = \{x \in R \mid 0 < x\}$ does not have a minimum element.
   (b) Show that if y is a minimum element for a set S of real numbers, then y is the infimum of S.