**Math 325-001 — Problem Set #10**
**Due: Monday, April 26 by midnight**

**Instructions:** You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) For each of the following, either give an example of a function with the indicated property or explain why none exists.
  (a) A function $f : (-1, 1) \to \mathbb{R}$ that is continuous at every point of $(-1, 1)$ whose range is $[1, 2)$.
  (b) A function $f : [-1, 1] \to \mathbb{R}$ that is continuous on the closed interval $[-1, 1]$ and whose range is $[1, 2)$.
  (c) A function $f : (-1, 1) \to \mathbb{R}$ that achieves a minimum but not a maximum value on $(-1, 1)$.
  (d) A function $f : [-1, 1] \to \mathbb{R}$ that is continuous on the closed interval $[-1, 1]$ with range $[0, \infty)$.
  (e) A function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every $x \in \mathbb{R}$ with range $(0, \infty)$.

(2) Find the derivative of $f(x) = \frac{1}{x}$ at any nonzero real number $r$ using just the definition of derivative.

(3) This problem illustrates that the derivative of a differentiable function need not be differentiable.[1] Let

$$f(x) = \begin{cases} \frac{x^3}{|x|} & \text{if } x \neq 0 \text{ and} \\ 0, & \text{if } x = 0. \end{cases}$$

  (a) For any $x \neq 0$, show that $f(x)$ is differentiable at $x$ and find a formula (with justification of course) for $f'(x)$.
  (b) Show $f$ is differentiable at 0 and find $f'(0)$.
  (c) Parts (a) and (b) show that $f'(x)$ is defined on all of $\mathbb{R}$. Prove $f'(x)$ is not differentiable at 0.

(4) Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that $f$ is differentiable at $x = 0$, and that $f$ is not differentiable at any other value of $x$.

[1] Note that in class we have seen (or will have seen) an example where a derivative is not even continuous.

(5) Assume $f(x)$ is continuous on the closed interval $[a.b]$ and that it is also one-to-one on $[a, b]$. (The latter term means that if $x, y$ are any two points in $[a, b]$ such that $x \neq y$, then $f(x) \neq f(y)$. Loosely: "no two distinct inputs have the same output".)

(a) Prove[2] that if $f(a) < f(b)$, then $f(a) < f(x) < f(b)$ for all $x$ such that $a < x < b$. *Tip:* Proceed by contradiction, and apply the Intermediate Value Theorem.

(b) Prove[3] that if $f(a) < f(b)$, then $f$ is strictly increasing on $[a, b]$; that is, for all $x$ and and $y$ such that $a \leq x < y \leq b$, we have $f(x) < f(y)$.

[2] Hint: Proceed by contradiction and use the Intermediate Value Theorem.

[3] Hint: Apply part (a) to the interval $[a, y]$.