Differential operators exercises

(1) Show that $\mathbb{C}\begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$ has no nonconstant $\mathrm{SL}_2(\mathbb{C})$-invariants of multidegree $(d, 0, \ldots, 0)$.

(2) If R is an A-algebra (any commutative rings), then $\mathrm{gr}(D^{\bullet}_{R|A})$ is commutative.

(3) If R is an A-algebra (any commutative rings), and $W \subset R$ is a multiplicative set, then $P^n_{W^{-1}R|A} \cong P^n_{W^{-1}R|(W\cap A)^{-1}A}$ and $D^n_{W^{-1}R|A} \cong D^n_{W^{-1}R|(W\cap A)^{-1}A}$.

(4) If S is a polynomial ring over A, and I an ideal of S, then $\{\delta \in D^n_{S|A} \mid \delta(S) \subseteq I\} = ID^n_{S|A}$.

(5) If S is a polynomial ring over A, then $D_{S|A}$ is a finitely generated A-algebra if and only if $\mathbb{Q} \subseteq A$.

(6) Show that $\mathbb{F}_p(t)[x]_{(x^p-t)}$ does not have any quasicoefficient field, and find a coefficient field for its completion.

(7) Let $(R, \mathfrak{m}, k)$ be a local K-algebra, and $\gamma \in R$ with image $\lambda \in k$. Let $f(T) \in K[T]$. Show that

$$1 \otimes f(\gamma) - f(\lambda) \otimes 1 \in (1 \otimes \gamma - \lambda \otimes 1) \cdot (k \otimes_K R).$$

(8) Let R be a local ring that is module-finite over a coefficient field K. Show that $D_{R|K} = \mathrm{Hom}_K(R, R)$.

(9) Let $R = \mathbb{C}[x^2, x^3] \subset S = \mathbb{C}[x]$. Show that, for $t < -2$, the vector space $[D_{R|\mathbb{C}}]_t / \overline{x^2}[D_{S|K}]_{t-2}$ is two-dimensional.

(10) Let G be a finite group acting linearly on a polynomial ring R over an algebraically closed field K. Let $\mathfrak{n}$ be a maximal ideal of R such that $g(\mathfrak{n}) \neq \mathfrak{n}$ for all $g \in G \smallsetminus \{e\}$, and let $\mathfrak{m} = \mathfrak{n} \cap R^G$. Then $\mathfrak{m}R = \bigcap_{g\in G} g(\mathfrak{n})$.

(11) If $R \to S$ is a map of A-algebras, and $\alpha_n : S \otimes_R P^n_{R|A} \to P^n_{S|A}$ is the natural map, then the composition

$$D^n_{S|A} \cong \mathrm{Hom}_S(P^n_{S|A}, S) \xrightarrow{\mathrm{Hom}(\alpha_n, S)} \mathrm{Hom}_S(S \otimes_R P^n_{R|A}, S) \cong \mathrm{Hom}_R(P^n_{R|A}, S) \cong D^n_{R|A}(R, S)$$

is just the restriction map.

(12) Let R be a polynomial ring over a field K of characteristic zero. Let G act on R linearly by

$$g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{for } g \in G,\ A \text{ an } n \times n \text{ matrix.}$$

Then the action of G on $D_{R|K}$ by conjugation is given by

$$g\begin{pmatrix}\overline{x_1}\\ \vdots\\ \overline{x_n}\\ \frac{\partial}{\partial x_1}\\ \vdots\\ \frac{\partial}{\partial x_n}\end{pmatrix} = \begin{bmatrix}A & 0\\ 0 & (A^T)^{-1}\end{bmatrix}\begin{pmatrix}\overline{x_1}\\ \vdots\\ \overline{x_n}\\ \frac{\partial}{\partial x_1}\\ \vdots\\ \frac{\partial}{\partial x_n}\end{pmatrix} \qquad \text{for } g \in G.$$

(13) Let p be a prime number. Show that

$$\binom{a}{p^e-1} \equiv \begin{cases} 1 \mod p & \text{if } a \equiv p^e - 1 \mod p^e \\ 0 \mod p & \text{otherwise.}\end{cases}$$

(14) Let $\overline{D^1_{R|A}}$ be the kernel of the map $D^1_{R|A} \to R$ given by evaluation at 1. Show that $\overline{D^1_{R|A}}$ is the module of A-linear maps from R to R that satisfy the Leibniz rule $\partial(xy) = x\partial(y) + y\partial(x)$.

(15) Let $R = A[x_1, \dots, x_n]$ be a polynomial ring over some ring A, and

$$\eta = \left[\frac{1}{x_1 \cdots x_n}\right] \in \frac{R_{x_1\cdots x_n}}{\sum_{i=1}^n R_{x_1\cdots\widehat{x_i}\cdots x_n}} = H^n_{(x_1,\dots,x_n)}(R).$$

Show that the annihilator of η in $D_{R|A}$ is the left ideal of $D_{R|A}$ generated by $(x_1, \dots, x_n)$.

(16) Check that if A is any commutative ring, and R is a polynomial ring over A, graded with $[R]_0 = A$ and each variable has degree one, that the restriction map

$$D^i_{R|A} \xrightarrow{\text{res}} \operatorname{Hom}_A([R]_{\leq i}, R)$$

is bijective.

(17) Let $A \to R \to S$ be commutative rings and M, N two S-modules. Show that

$$D^i_{S|R}(M,N) \subseteq D^i_{S|A}(M,N) \subseteq D^i_{R|A}(M,N).$$

(18) For a polynomial ring over a field, we have

$$[\overline{x^\alpha}\partial^{(\beta)}, \overline{x_i}] = \overline{x^\alpha}\partial^{(\beta-e_i)} \qquad \text{and} \qquad [\overline{x^\alpha}\partial^{(\beta)}, \partial^{(e_i)}] = -\alpha_i\overline{x^{\alpha-e_i}}\partial^{(\beta)}.$$

(19) Show that if K is a field, $R = K[x_1, \dots, x_n]$ is a polynomial ring, I is an ideal of R, and i is a positive integer, then $H^i_I(R)$ is either zero or a faithful R-module, by using the fact that this is true whenever K is a perfect field.

(20) Let $R = \mathbb{C}[y] \to S = \mathbb{C}[x]$ via $y \mapsto x^2$. Show that the differential operator $\frac{\partial}{\partial y}$ on R does not extend to a differential operator on S.

(21) Show that, for $R = \mathbb{C}[x,y]/(xy)$, $\operatorname{gr}^{\text{ord}}(D_{R|\mathbb{C}})$ is not Noetherian.

(22) Show that, for $R = \mathbb{C}[x,y]/(xy)$, $D_{R|\mathbb{C}}$ is both left and right Noetherian.