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Chapter 0

Introduction

This class is on differential operator (both smooth and singular setting) and various applications to Commutative Algebra.

Basic examples of a differential operator on polynomial rings $K[x_1, \cdots, x_n]$ over afield K, $\delta = \frac{\partial}{\partial x_1}$ is a differential operator.

K does not need to be $\mathbb{R}$ or $\mathbb{C}$ use linearity and product/power rule instead of analogous.

In general, given a A-algebra R (A, R commute) we will have filtered noncommutative ring of differential operators:

$$\begin{array}{rcl} A \longrightarrow R & \rightsquigarrow & \mathrm{D}_{R/A} \\ & & \mathrm{D}^0_{R/A} \subseteq \mathrm{D}^1_{R/A} \subseteq \mathrm{D}^2_{R/A} \subseteq \dots \end{array}$$

The element operator on R:

$$\mathrm{D}_{R/A} \curvearrowright R,$$

R is a $\mathrm{D}_{R/A}$-module.

This action is a way to undo multiplication, (or at least to try to!) in a way that preserve structure e.g in $R[x]$, multiplication by $x^n : 1 \longrightarrow x^n$ and no R-operation can send $x^n \longrightarrow 1$, but in $\mathrm{D}_{R/k}$, we have $\left(\frac{\partial}{\partial x}\right)^n$

$$1 \xrightarrow{\cdot x^n} x^n \xrightarrow{\cdot \frac{\partial}{\partial x}} n! \xrightarrow{1/n!} 1,$$

and $\left(\frac{\partial}{\partial x}\right)^n$ is not a random function sending $x^n \longrightarrow n!$; it is a function we understand well.

More algebraically, $1 \notin R \cdot x^n$, but $1 \in \mathrm{D}_{R/K} \cdot x^n$. Along similar lines, R is not a simple R-module, but R is a simple $\mathrm{D}_{R/K}$-module. Then, R-module often become small $\mathrm{D}_{R/K}$-modules.

Will develop general theory of differential operators for algebras $A \longrightarrow R$. In general, it is hard to compute the rings $\mathrm{D}_{R/A}$ and to understand what good properties they have or do not have. At least when $R = A[x_1, \dots, x_n]$ polynomial ring, can compute and establish many good properties.

If also we assume A is a field of characteristic zero, can say extremely strong and specialized properties of $\mathrm{D}_{R/A}$. There is a lot of research on this that touches many fields of math, and many extremely deep results and complicated machinery. Will develop just a bit of this at and of does, but focus more on more general situation.

What applications?

Symbolic powers

Given a prime ideal $P \subseteq R$, we have $P^{(n)} := P^n R \cap R$. Will give a very concrete characterization of $P^{(n)}$ in poly ring R, and a characterization of $P^{(n)}$ more generally, in terms of differential operators.

Singularities

A local ring $(R, \mathfrak{m}, K)$ satisfies the inequality $\dim R \leq \dim(\mathfrak{m}/\mathfrak{m}^2)$ (equal minimal number of generators) R is regular if the equal holds, and singular otherwise. Will use properties of $\mathrm{D}_{R/A}$ to characterize when R is regular, and various classes of singularities.

Local Cohomolgy

Given $(f_1, \dots, f_n) = I$ ideal in R, we define local cohomology modules of R with support in I as

$$H^i_I(R) = H^i(0 \longrightarrow R \longrightarrow \bigoplus_i R_{f_i} \longrightarrow \bigoplus_{i<j} R_{f_i f_j} \longrightarrow \cdots \longrightarrow R_{f_1 \cdots f_n} \longrightarrow 0),$$

With map as $\pm 1's$ in each component in such away as to make a complex. The modules R_{f_i} and $H^i_I(R)$ are usually not finitely generated R-modules. When R is a polynomial ring over field of characteristic zero, will find that R_{f_i}, $H^i_I(R)$ have finite length as $\mathrm{D}_{R/K}$-modules, and finitely many associated primes as R-modules.

Chapter 1

Basics of differential operators

1.1 Differential operators on polynomial rings in characteristic zero

We will give a definition of differential operators in general, but we first give a special definition in the most special case.

Definition 1.1.1. Let K be a field of characteristic zero, and $R = K[x_1, \ldots, x_n]$ be a polynomial ring. The ring of K-linear differential operators on $R$is the subring of $\mathrm{End}_K(R)$ (K-linear endomorphisms of R with composition as multiplication) generated by

- K,
- $\overline{x}_1, \ldots, \overline{x}_n$ (the endomorphisms $\overline{x}_i : f \longrightarrow x_i \cdot f$),
- $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, where $\frac{\partial}{\partial x_i}$ is the K-linear map given by $\frac{\partial}{\partial x_i}(x_1^{a_1} \cdots x_i^{a_i} \cdots x_n^{a_n}) = a_i x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_n^{a_n}$.

We write $\mathrm{D}_{R/K}$ for this, $\mathrm{D}_{R/K} = K \left\langle \overline{x}_1, \ldots, \overline{x}_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle$.

Observation 1.1.2.

(1) Since $\mathrm{D}_{R/K} \subseteq \mathrm{End}_K(R)$, every element of $\mathrm{D}_{R/K}$ is a K-vector space endomorphism of R. This make R into a left $\mathrm{D}_{R/K}$-module in a canonical way, where the module action is "apply the endomorphism." Let's umpackagen this once just so were one hundred procent sure that we want "left" instead of "right."

Let $\alpha, \beta \in \mathrm{D}_{R/K}, r \in R$.

$$\begin{aligned}(\alpha \cdot \beta) \cdot r &:= (\alpha \circ \beta)(r) \\ &= \alpha(\beta(r)) \\ &= \alpha(\beta \cdot r) \\ &= \alpha \cdot (\beta \cdot r),\end{aligned}$$

and

$$\begin{aligned}(\alpha + \beta) \cdot r &:= (\alpha + \beta)(r) \\ &= \alpha(r) + \beta(r) \\ &= (\alpha \cdot r) + (\beta \cdot r).\end{aligned}$$

Where "$\cdot$" is the $\mathrm{D}_{R/K}$-multiplication and "$\cdot$" is the $\mathrm{D}_{R/K} \curvearrowright R$-action. We will use both module action notation and function notation.

(2) We have $K\langle \overline{x}_1, \ldots, \overline{x}_n \rangle \subseteq \mathrm{D}_{R/K}$ by definition. Each map $\overline{x_i}$ is in fact R-linear, so $K\langle \overline{x}_1, \ldots, \overline{x}_n \rangle \subseteq \mathrm{End}_K(R)$, which is equal to $\mathrm{Hom}_R(R, R) \cong R$ as an R-module.

In fact, $K\langle \overline{x}_1, \ldots, \overline{x}_n \rangle \cong K[x_1, \ldots, x_n] = R$ as rings by the obvious map $(x_i \longrightarrow \overline{x}_i)$. Thus, there is an injective K-algebra homomorphism

$$R \hookrightarrow \mathrm{D}_{R/K}$$

sending r to "multiplication by r".

We should exercise some caution in distinguishing elements in R from their images in $\mathrm{D}_{R/K}$ (the multiplications). For example,

$$\left(\frac{\partial}{\partial x}\right)^2 \cdot x = 0,$$

where "$\cdot$" is the $\mathrm{D}_{R/K} \curvearrowright R$-action. But,

$$\left(\frac{\partial}{\partial x}\right)^2 \cdot \overline{x} \neq 0,$$

Since $\left(\frac{\partial}{\partial x}\right)^2 \cdot \overline{x}(x) = \left(\frac{\partial}{\partial x}\right)^2 \cdot (x^2) = 2$, so $\left(\frac{\partial}{\partial x}\right)^2 \cdot \overline{x}$ is not zero endomorphism of R. Where "$\cdot$" is the $\mathrm{D}_{R/K}$-multiplication.

We will not worry so much for elements of K, since they commute with everything.

1.2 Relations and standard forms

To try to understand $\mathrm{D}_{R/K}$, we want to find relations between its generators. Some generators commute:

$$\begin{cases} \overline{x}_i\overline{x}_j = \overline{x}_j\overline{x}_i & \text{if } i \neq j \\ \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} & \text{if } i \neq j \\ \overline{x}_i\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\overline{x}_i & \text{if } i \neq j. \end{cases} \tag{1.2.0.1}$$

Indeed, it suffices to check these on a K-vector space basis for R, e.g. the monomials, where it is clear.

We now need to see how variables and partial derivatives commute or fail to we have

$$\begin{aligned} \frac{\partial}{\partial x_i}\overline{x}_i(x_1^{a_1}\cdots x_i^{a_i}\cdots x_n^{a_n}) &= \frac{\partial}{\partial x_i}(x_1^{a_1}\cdots x_i^{a_i+1}\cdots x_n^{a_n}) \\ &= (a_i+1)(x_1^{a_1}\cdots x_i^{a_i}\cdots x_n^{a_n}), \end{aligned}$$

so $\frac{\partial}{\partial x_i}\overline{x}_i = \overline{a_i+1}$, whereas

$$\begin{aligned} \overline{x}_i\frac{\partial}{\partial x_i}(x_1^{a_1}\cdots x_i^{a_i}\cdots x_n^{a_n}) &= \overline{x}_i(a_ix_1^{a_1}\cdots x_i^{a_i-1}\cdots x_n^{a_n}) \\ &= a_ix_1^{a_1}\cdots x_i^{a_i}\cdots x_n^{a_n}, \end{aligned}$$

so $\overline{x}_i\frac{\partial}{\partial x_i} = \overline{a_i}$.

In particular,

$$\frac{\partial}{\partial x_i}\overline{x}_i = \overline{x}_i\frac{\partial}{\partial x_i} + \overline{1} \tag{1.2.0.2}$$

for each i.

1.3 Application: Invariants of classical groups

Perhaps the first application historically of differential operators in commutative algebra is to computation of invariant rings. This goes back to 19^{th} century (Gordam, Cayley, Hilbert, etc, and was recapped nicely in Weyl's **Classical Groups**.

We do not need to know much about differential generators; this is a bones applications of the philosophy that differential operators undo multiplication/ decrease order in a structured way.

We will let K be a field of characteristic zero,and $R = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$ be a polynomial ring in $2n$ variables. For shrt, we will write $\underline{x}(x_1,\ldots,x_n)$ and $\underline{y}(y_1,\ldots,y_n)$.

1.4 Some particular differential operators

- The **Euler operator** on our polynomial ring is

$$\mathrm{E} = \sum_i \overline{x}_i \frac{\partial}{\partial x_i} + \sum_i \overline{y}_i \frac{\partial}{\partial y_i},$$

  and likewise on any polynomial ring. The point is **Euler's identity**: if f is homogeneous of degree d, we have $\mathrm{E}(f) = df$.

  To prove it, we check for monomials:

$$\begin{aligned}\mathrm{E}(x_1^{\alpha_1} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_n^{\beta_n}) &= \sum_i \alpha_i \underline{x}^{\underline{\alpha}} \underline{y}^{\underline{\beta}} + \sum_i \beta_i \underline{x}^{\underline{\alpha}} \underline{y}^{\underline{\beta}} \\ &= \left(\sum_i \alpha_i + \sum_i \beta_i \right) (\underline{x}^{\underline{\alpha}} \underline{y}^{\underline{\beta}}).\end{aligned}$$

In our situation, we have finer gradings that elate to other operators. We give R an $\mathbb{N}^n$-grading by setting $|\overline{x}_i| = |\overline{y}_i| = (0, \ldots, 0, 1, 0, \ldots, 0) =: e_i$ for each i.

- Set $\mathrm{E}_{ii} = \overline{x}_i \frac{\partial}{\partial x_i} + \overline{y}_i \frac{\partial}{\partial y_i}$ for each i. If f is homogeneous in the $\mathbb{N}^n$-grading, with degree $(d_1, \ldots, d_n)$, then $\mathrm{E}_{ii}(f) = d_i \cdot f$. Check in the same way as above.

- Set $\mathrm{E}_{ij} = \overline{x}_i \frac{\partial}{\partial x_j} + \overline{y}_i \frac{\partial}{\partial y_j}$ for $i \neq j$. This is called a polarization operator. Intuitively, this turns j variables into i variables; more precisely, if

$$|f| = (d_1, \ldots, d_i, \ldots, d_j, \ldots, d_n),$$

  then

$$|\mathrm{E}_{ij}(f)| = (d_1, \ldots, d_i + 1, \ldots, d_j - 1, \ldots, d_n).$$

- Set $\mathrm{D}_{ij} = \det \begin{pmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \end{pmatrix} = \frac{\partial}{\partial x_i} \circ \frac{\partial}{\partial y_j} - \frac{\partial}{\partial y_i} \circ \frac{\partial}{\partial x_j}$ for $i \neq j$.

We will need the observation that E_{ii} and E_{ij} are **derivations**: K-linear map δ that satisfy $\delta(fg) = f\delta(g) + g\delta(f)$ for $f, g \in R$. Indeed, $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ all satisfy this product rule, and any $\overline{R}$-linear combination of derivations is a derivation.

1.5 Capelli's identity

Theorem 1.5.1 (Capelli)**.** $\det\left(\begin{bmatrix} \mathrm{E}_{ii}+\overline{1} & \mathrm{E}_{ij} \\ \mathrm{E}_{ji} & \mathrm{E}_{jj} \end{bmatrix}\right) = \det\left(\begin{bmatrix} \overline{x}_i & \overline{x}_j \\ \overline{y}_i & \overline{y}_j \end{bmatrix}\right) \circ \det\left(\begin{bmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \end{bmatrix}\right).$

Proof.

$$\begin{aligned}
\text{LHS} &= \mathrm{E}_{ii}+\overline{1}\circ \mathrm{E}_{jj} - \mathrm{E}_{ji}\circ \mathrm{E}_{ij} \\
&= \left(\overline{x}_i\frac{\partial}{\partial x_i} + \overline{y}_i\frac{\partial}{\partial y_i} + \overline{1}\right)\circ\left(\overline{x}_j\frac{\partial}{\partial x_j} + \overline{y}_j\frac{\partial}{\partial y_j}\right) \text{ all terms commute indistributing} \\
&- \left(\overline{x}_j\frac{\partial}{\partial x_i} + \overline{y}_j\frac{\partial}{\partial y_i}\right)\circ\left(\overline{x}_i\frac{\partial}{\partial x_j} + \overline{y}_i\frac{\partial}{\partial y_j}\right) \text{ not all terms commute} \\
&= \overline{x}_j\overline{y}_i\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial y_i} - \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}\right) + \overline{x}_i\overline{y}_j\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial y_i}\right) \\
&= \text{RHS}
\end{aligned}$$

□

1.6 SL_2-action

Let $G = SL_2(K)$ action on R by

$$g\cdot\begin{bmatrix}\underline{x}\\ \underline{y}\end{bmatrix} = g\begin{bmatrix}\underline{x}\\ \underline{y}\end{bmatrix} \text{ matrix product}$$

i.e,

$$g = \begin{bmatrix} a & b \\ c & d\end{bmatrix} \rightsquigarrow g\begin{bmatrix}\underline{x}\\ \underline{y}\end{bmatrix} = \begin{bmatrix} a\underline{x}+b\underline{y} \\ c\underline{x}+d\underline{y}\end{bmatrix},$$

so

$$\begin{aligned} g(x_i) &= ax_i + by_i, \\ g(y_i) &= cx_i + dy_i. \end{aligned}$$

We want to compute all the invariants of this action. Set

$$\overline{\Delta}_{ij} = \det\left(\begin{bmatrix} \overline{x}_i & \overline{x}_j \\ \overline{y}_i & \overline{y}_j \end{bmatrix}\right)$$

and

$$\Delta_{ij} = \det\left(\begin{bmatrix} x_i & x_j \\ y_i & y_j \end{bmatrix}\right)$$

Each Δ_{ij} is an invariant, since multiplying by a det -1 matrix does not affect the determinate of this submatrix. Thus, $S := K[\{\Delta_i \mid i < j\}] \subseteq R^G$ is a subring of the invariant ring.

We will show that actually $S = R^G$.

Lemma 1.6.1. *S and R^G are $\mathbb{N}^n$-graded subrings of R.*

Proof. For S, it is clear since its generators are homogeneous. For R^G, we note that the action of G preserves degrees in the $\mathbb{N}^n$-grating, so the lemma follows. □

Lemma 1.6.2. *We have:*

(i) $\mathrm{E}_{ij}(R^G) \subseteq R^G$, *if* $i \leq j$;

(ii) $\mathrm{E}_{ij}(S) \subseteq S$, *if* $i \leq j$;

(iii) $\mathrm{D}_{ij}(R^G) \subseteq R^G$, *if* $i < j$.

Proof. (i) First, we shoe that $\mathrm{E}_{ij} \circ g = g \circ \mathrm{E}_{ij}$ for $g \in SL_2(K)$. Write $\underline{x}' = a\underline{x} + b\underline{y}$ and $\underline{y}' = c\underline{x} + d\underline{y}$. We have

$$\begin{aligned} = (g \circ \mathrm{E}_{ij})(f(\underline{x}, \underline{y})) = g\left(x_i \frac{\partial f}{\partial x_j}(\underline{x}, \underline{y}) + y_i \frac{\partial f}{\partial y_j}(\underline{x}, \underline{y})\right) \\ = x_i' \frac{\partial f}{\partial x_j}(\underline{x}', \underline{y}') + y_i' \frac{\partial f}{\partial y_j}(\underline{x}', \underline{y}'), \end{aligned}$$

and

$$\begin{aligned} (\mathrm{E}_{ij} \circ g)(f(\underline{x}, \underline{y})) &= x_i \frac{\partial f}{\partial x_j}(\underline{x}', \underline{y}') + y_i \frac{\partial f}{\partial y_j}(\underline{x}', \underline{y}') \\ &= (ax_i + by_i) \frac{\partial f}{\partial x_j}(\underline{x}', \underline{y}') + (cx_i + dy_i) \frac{\partial f}{\partial y_j}(\underline{x}', \underline{y}') \\ &= x_i' \frac{\partial f}{\partial x_j}(\underline{x}', \underline{y}') + y_i' \frac{\partial f}{\partial y_j}(\underline{x}', \underline{y}'). \end{aligned}$$

(iii) Similar

(ii) We use that $\mathrm{E}_{ij}^{'s}$ are derivations. First, we compute $\mathrm{E}_{ij}(\Delta_{k\ell}) \in \{0, \Delta_{ik}, \pm 2\}$ for various cases so $\mathrm{E}_{ij}(\Delta_{k\ell}) \in S$.

Now, use the product rule to conclude that E_{ij} of any product of these is again in S.

□

Lemma 1.6.3. *There are no (nonconstant) homogeneous of degree $(d, 0, \ldots, 0)$.*

Theorem 1.6.4. *$R^{SL_2} = K[\{\Delta_{ij}\}]$. That is, $S = R^G$.*

Proof. It suffices to show that any homogeneous element of R^G is in S. If not, we pick $f \in R^G \backslash S$ homogeneous of degree $(d_1, \ldots, d_n)$ of minimal total degree ($d_1 + \cdots + d_n = d$ minimal), and among these, with d_1 maximal. We know $d_1 \neq d$ by Lemma 1.6.3, so $d_j \neq 0$ for some j. Then,

$$((\mathrm{E}_{11} + \overline{1}) \circ \mathrm{E}_{jj})(f) = \mathrm{E}_{j1} \circ \mathrm{E}_{1j}(f) + \Delta_{1j} \circ \mathrm{D}_{1j}(f).$$

We have $\mathrm{D}_{1j}(f) \in R^G$, and f of minimal degree not in S, then $f \in S$, and $\Delta_{ij} \in S$. $((\mathrm{E}_{11} + \overline{1}) \circ \mathrm{E}_{jj})(f) = (d_1 + 1)(d_j)f$ is a nonzero scalar multiple of f.

The first component of degree of $\mathrm{E}_{1j}(f)$ is greater than that of f, and total degree is same, so $\mathrm{E}_{1j}(f) \in S$ (because in R^G), and $(\mathrm{E}_{j1} \circ \mathrm{E}_{1j})(f) \in S$ by Lemma 1.6.2. Thus, LHS $\in S$, so $f \in S$. □

1.7 A bit more about differential operators on the polynomial ring

Let K be a field of characteristic zero, $R = [x_1, \ldots, x_n]$ be polynomial ring, and $\mathrm{D}_{R/K} = K\left\langle \overline{x}_1, \ldots, \overline{x}_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle \subseteq \mathrm{End}_K(R)$.

Definition 1.7.1. Given two endomorphisms α, β of an abelian group A, we write $[\alpha, \beta] := \alpha \circ \beta - \beta \circ \alpha \in \mathrm{End}(A)$.

From relations 1.2.0.1 and 1.2.0.2, we have

$$[\overline{x}_i, \overline{x}_j] = [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_J}] = [\overline{x}_i, \frac{\partial}{\partial x_J}] = 0$$
$$[\frac{\partial}{\partial x_i}, \overline{x}_i] = 1,$$

in $\mathrm{D}_{R/K}$.

Want to use these relations to write any differential operator in a standard form recognize when two operators are same or different.

Lemma 1.7.2. *i)* $\left(\frac{\partial}{\partial x_i}\right)^a \overline{x}_i = \overline{x}_i \left(\frac{\partial}{\partial x_i}\right)^a + a \left(\frac{\partial}{\partial x_i}\right)^{a-1}$,

ii) $\left(\frac{\partial}{\partial x_i}\right)^a \overline{x}_i^b = \sum_{k=0}^{\min\{a,b\}} k! \binom{a}{k} \binom{b}{k} \overline{x}_i^{b-k} \left(\frac{\partial}{\partial x_i}\right)^{a-k}$

Proof. 1) By induction on a, with $a = 1$ already done.

Induction step:

$$\begin{aligned}\left(\frac{\partial}{\partial x_i}\right)^a \overline{x}_i &= \left(\frac{\partial}{\partial x_i}\right)\left(\frac{\partial}{\partial x_i}\right)^{a-1} \overline{x}_i \\ &= \frac{\partial}{\partial x_i}\left(\overline{x}_i \left(\frac{\partial}{\partial x_i}\right)^{a-1} + (a-1)\left(\frac{\partial}{\partial x_i}\right)^{a-2}\right) \\ &= \left(\overline{x}_i \frac{\partial}{\partial x_i} + \overline{1}\right)\left(\frac{\partial}{\partial x_i}\right)^{a-1} + (a-1)\left(\frac{\partial}{\partial x_i}\right)^{a-1} \\ &= \overline{x}_i \left(\frac{\partial}{\partial x_i}\right)^a + a\left(\frac{\partial}{\partial x_i}\right)^{a-1}.\end{aligned}$$

ii) Similar, but messy.

□

Rather than the precise form of (ii), we will mostly care about this as saying we can switch the order and write as same product "reversed" (note first coefficient is 1) plus smaller terms.

Proposition 1.7.3. *Any element in $\delta \in \mathrm{D}_{R/K}$ can be write as*

$$\delta = \sum_{\alpha_1,\ldots,\alpha_n} \overline{\gamma}_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

for some $\gamma_\alpha'^s$. That is, $\mathrm{D}_{R/K}$ is generated by $\left\{\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}\right\}$ as an (left) $\overline{R}$-module, where $\overline{R}$ is the image of R in $\mathrm{D}_{R/K}$.

Proof. Using the commutation relation, we can express any element as a sum of products of the form $\overline{x}_1^{a_{11}}\left(\frac{\partial}{\partial x_1}\right)^{b_{11}} \overline{x}_1^{a_{12}}\left(\frac{\partial}{\partial x_1}\right)^{b_{12}} \cdots \overline{x}_1^{a_{1j}}\left(\frac{\partial}{\partial x_1}\right)^{b_{1j}}$, and similar terms in other indices. Apply Lemma 1.7.2 to "straighten out" $\overline{x}_1^{a_{1j}}\left(\frac{\partial}{\partial x_1}\right)^{b_{1j-1}}$ as a sum of products with $\overline{x}'^s$ before $\frac{\partial}{\partial x}'^s$. Inductively, we obtain elements of desired form. □

Theorem 1.7.4. *The expressions given in Proposition 1.7.3 are unique. That is, $\left\{\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}\right\}$ is a free basis for $\mathrm{D}_{R/K}$ as a (left) $\overline{R}$-module.*

Proof. Suppose that $\delta = \sum_\alpha \overline{\gamma}_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ is the zero operator. We need to see that each $\overline{\gamma}_\alpha$ is zero. If not, pick a tuple β with $\overline{\gamma}_\beta \neq 0$, and $\beta_1 + \cdots + \beta_n$ minimal among such tuples. We compute $\delta(x_1^{\beta_1}\cdots x_n^{\beta_n})$. Since $\delta = 0$, $\delta(x_1^{\beta_1}\cdots x_n^{\beta_n}) = 0$. If $\alpha_1 + \cdots + \alpha_n \geq \beta_1 + \cdots + \beta_n$, then we either have $\alpha_i = \beta_i$ for each i ($\alpha = \beta$) or $\alpha_i > \beta_i$ for some i. Thus, if $\overline{\gamma}_\alpha \neq 0$ and $\alpha \neq \beta$, we have $\overline{\gamma}_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} (x_1^{\beta_1}\cdots x_n^{\beta_n}) = 0$, so

$$\begin{aligned}
\delta(x_1^{\beta_1}\cdots x_n^{\beta_n}) &= \sum_\alpha \overline{\gamma}_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} x_1^{\beta_1}\cdots x_n^{\beta_n} \\
&= \overline{\gamma}_\beta \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n} x_1^{\beta_1}\cdots x_n^{\beta_n} \\
&= \overline{\gamma}_\beta(\beta_1!\cdots\beta_n!) \\
&= \beta_1!\cdots\beta_n!\overline{\gamma}_\beta.
\end{aligned}$$

Thus, $\overline{\gamma}_\beta = 0$, contradicting choice of $\overline{\gamma}_\beta \neq 0$, so we must have $\delta = 0$. □

Remark 1.7.5. Characteristic zero was used in an important way here. In characteristic p, we have $\left(\frac{\partial}{\partial x_i}\right)^p = 0$.

1.8 Grading on $\mathrm{D}_{R/K}$

An element of $\mathrm{D}_{R/K}$ is **homogeneous** of degree d if $\delta(R_i) = R_{i+d}$ for each i. The ring $\mathrm{D}_{R/K}$ is **graded** in this way any element is uniquely a sum of homogeneous elements, and product of homogeneous elements are homogeneous of degree equal to the sum of the degrees. E.g., $\deg\left(\frac{\partial}{\partial x_i}\right) = -1$, $\deg\left(\overline{x}_1^3 \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} + \overline{x}_3\right) = 1$.

1.9 Order filtration

We define an ascending filtration on $\mathrm{D}_{R/K}$ by

$$\mathrm{D}^i_{R/K} = \bigoplus_{\alpha_1+\cdots+\alpha_n\leq i} \overline{R}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

these are the **elements of order at most** i. We observe that this **multiplicative** $\mathrm{D}^i_{R/K}\cdot\mathrm{D}^j_{R/K} \subseteq D^{i+j}_{R/K}$ for each i, j. If suffices to check for "monomials"

$$\left(\bar{x}_1^{a_1}\cdots\bar{x}_n^{a_n}\frac{\partial}{\partial x_1}^{b_1}\cdots\frac{\partial}{\partial x_n}^{b_n}\right)\left(\bar{x}_1^{c_1}\cdots\bar{x}_n^{c_n}\frac{\partial}{\partial x_1}^{d_1}\cdots\frac{\partial}{\partial x_n}^{d_n}\right)$$

with $b_1 + \cdots + b_n \leq i$ and $d_1 + \cdots + d_n \leq j$. Can write $\frac{\partial}{\partial x_1}^{b_1}\bar{x}_1^{c_1}$ as a sum in standard form with partials to the $\leq b_1$; similarly for each i.

1.10 Associated graded at order filtration

Given a multiplicative ascending filtration

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

of a ring T, the **associated graded ring** of the filtration is

$$gr_{F_\bullet}(T) = \sum_{i \geq 0} F_i / F_{i-1} \quad (F_{-1} := 0)$$

as graded abelian group, with multiplication $(f + F_{i-1})(g + F_{j-1}) = (fg + F_{1+j-1})$ where $f \in F_i$ $g \in F_j$, and $fg \in F_{i+j}$; this is well-defined by multiplicative hypothesis.

We compute this for the ring f differential operators on poly ring in characteristic zero, with order filtration we have

$$gr^{\mathrm{ord}}(\mathrm{D}_{R/K})_i = \frac{\mathrm{D}^i_{R/K}}{\mathrm{D}^{i-1}_{R/K}} \cong \sum_{\alpha_1 + \cdots + \alpha_n = i} \bar{R} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

. We observe that

$$\left(\bar{r}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} + \mathrm{D}^{i-1}_{R/K}\right)\left(\bar{s}\left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n} + \mathrm{D}^{j-1}_{R/K}\right)$$
$$= \bar{r}\bar{s}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1+\beta_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n+\beta_n} + \mathrm{D}^{i+j-1}_{R/K} :$$

use that $\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} x_1^{c_1} = x_1^{c_1}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}$ + lower order terms... Thus, $gr^{\mathrm{ord}}(\mathrm{D}_{R/K})$ is commutative.

There is a ring homomorphism (K-algebra homomorphism)

$$\begin{aligned} K[x_1, \ldots, x_n, y_1, \ldots, y_n] &\longrightarrow gr^{\mathrm{ord}}(\mathrm{D}_{R/K}) \\ x_i &\longrightarrow \bar{x}_i \\ y_i &\longrightarrow \frac{\partial}{\partial x_i} + \mathrm{D}^0_{R/K}, \end{aligned}$$

and using the $\overline{R}$-module structure above, this is on isomorphism. In this way, we can think of $\mathrm{D}_{R/K}$ as close to a polynomial ring in $2n$ variables.

Chapter 2

Differential operators in general

Definition 2.0.1. Let, $A \longrightarrow R$ be a homomorphism at commutative rings. Let M and N be two R-modules. The differential operators of order i from M to N are defined inductively in i as follows:

$$\mathrm{D}^0_{R/A}(M,N) = \mathrm{Hom}_R(M,N)$$
$$\mathrm{D}^i_{R/A}(M,N) = \{\delta \in \mathrm{Hom}_A(M,N) \mid \delta \circ \overline{f_M} - \overline{f_N} \circ \delta \in \mathrm{D}^{i-1}_{R/A}(M,N) \text{ for all } f \in R\}$$

we define $\mathrm{D}_{R/A}(M,N) = \bigcup_{i\in\mathbb{N}} \mathrm{D}^i_{R/A}(M,N)$.

Remark 2.0.2. A function $\alpha \in \mathrm{Hom}_A(M,N)$ is R-linear (ie., $\alpha \in \mathrm{Hom}_R(M,N)$) if and only if $\delta \circ \overline{f_M} = \overline{f_N} \circ \delta$ for all $f \in R$. Thus, we get the same notion if we set $\mathrm{D}^{-1}_{R/A}(M,N) = 0$ and use save inductive step.

Notation 2.0.3. Given $\alpha, \beta \in \mathrm{End}_A(M)$ some A-module M, we write $[\alpha, \beta] := \alpha \circ \beta - \beta \circ \alpha \in \mathrm{End}_A(M)$. We also abuse this notation, e.g., by writing $[\overline{\alpha}, \overline{f}]$ for $\alpha \circ \overline{f_M} - \overline{f_N} \circ \alpha$ for $\alpha \in \mathrm{Hom}_A(M,N)$.

Proposition 2.0.4. *For α, β, γ homomorphisms of modules or "f", the following hold whenever defined:*

i) $[\alpha, \beta + \gamma] = [\alpha, \beta] + [\alpha, \gamma]$ *and* $[\alpha + \beta, \gamma] = [\alpha, \gamma] + [\beta, \gamma]$;

ii $[\overline{a}\alpha, \beta] = [\alpha, \overline{a}\beta] = \overline{a}[\alpha, \beta]$ *if* $a \in A$ *and* α, β *A-linear;*

iii) $[\alpha, \beta] = -[\beta, \alpha]$;

iv) $[\alpha\beta, \gamma] = \alpha[\beta, \gamma] + [\alpha, \gamma]\beta$ *and* $[\alpha, \beta\gamma] = [\alpha, \beta]\gamma + \beta[\alpha, \gamma]$;

v) $[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0$.

Proof. i) $[\alpha, \beta+\gamma] = \alpha(\beta+\gamma) - (\beta+\gamma)\alpha = \alpha\beta + \alpha\gamma - \beta\alpha - \gamma\alpha = \alpha\beta - \beta\alpha + \alpha\gamma - \gamma\alpha = [\alpha, \beta] + [\alpha, \gamma]$ and similarly etc.

v) LHS= $(\alpha\beta-\beta\alpha)\gamma-\gamma(\alpha\beta-\beta\alpha)+(\beta\gamma-\gamma\beta)\alpha-\alpha(\beta\gamma-\gamma\beta)+(\gamma\alpha-\alpha\gamma)\beta-\beta(\gamma\alpha-\alpha\gamma)$ all cancels.

□

Lemma 2.0.5. *If $r \in R$, $\alpha \in \mathrm{D}^i_{R/A}$, then $\overline{r}\alpha \in \mathrm{D}^i_{R/A}$.*

Proof. If $i = 0$, this is clear.

Let $f \in R$. We have that $[\overline{r}\alpha, \overline{f}] = \overline{r}[\alpha, \overline{f}] + [\overline{r}, \overline{f}]\alpha = \overline{r}[\alpha, \overline{f}]$, and it follows from induction. Similarly for $\alpha\overline{r}$. □

Proposition 2.0.6. *Let $R = A[f_1, \dots, f_n]$; i.e, R is generated by $\{f_1, \dots, f_n\}$ as A-algebra. Then, $\delta \in \mathrm{End}_A(R)$ is in $\mathrm{D}^i_{R/A}$ if and only if $[\delta, \overline{f}_j] \in \mathrm{D}^{i-1}_{R/A}$, $j = 1, \dots t$.*

Proof. By A-linearity, it suffices to show that the hypothesis implies $[\delta, \overline{f}_1^{a_1} \cdots \overline{f}_t^{a_t}] \in \mathrm{D}^{i-1}_{R/A}$. Write $f_1^{a_1} \cdots f_t^{a_t} = f_i \cdot \mu$, μ of "lower degree" in the f'^s. Then,

$$[\delta, \overline{f}_i\overline{\mu}] = [\delta, \overline{f}_i]\overline{\mu} + \overline{f}_i[\delta, \overline{\mu}] \in \mathrm{D}^{i-1}_{R/A}$$

inductively. □

We now generalize Lemma 2.0.5.

Proposition 2.0.7. *Let $A \longrightarrow R$ commutative rings, and L, M, N R-modules. The, $\alpha \in \mathrm{D}^i_{R/A}(M, N), \beta \in \mathrm{D}^j_{R/A}(L, M)$ implies $\alpha \circ \beta \in \mathrm{D}^{i+j}_{R/A}(L, N)$.*

Proof. If $i + j = 0$, this is clear.

Let $f \in R$. We have that $[\alpha \circ \beta, \overline{f}] = \alpha[\beta, \overline{f}] + [\alpha, \overline{f}]\beta$, and apply induction on $i + j$. □

Corollary 2.0.8. *If $A \longrightarrow R$ commutative rings, then $\mathrm{D}_{R/A} := \mathrm{D}_{R/A}(R, R)$ is a ring under composition. If M is an R-module, $\mathrm{D}_{R/A}(M, M)$ is a ring.*

Proof. As subset of $\mathrm{End}_A(R)$ or $\mathrm{End}_A(M)$, we just need to check these are closed under composition and subtraction (and have 1). □

Moreover, these are filtered rings, $\mathrm{D}^i_{R/A} \cdot \mathrm{D}^j_{R/A} \subseteq \mathrm{D}^{i+j}_{R/A}$, and likewise with (M, M).

We have an isomorphism $R \cong \mathrm{D}^0_{R/A}$ of rings, we will also write $\overline{R}$ for $\mathrm{D}^0_{R/A}$. The image of A is contained in the center of $\mathrm{D}_{R/A}$ by definition.

Proposition 2.0.9. *Let $R = A[f_1, \dots, f_t]$, and $\alpha, \beta \in \mathrm{D}^i_{R/A}$. Then, $\alpha = \beta$ if and only if $\alpha(f_1^{c_1} \cdots f_t^{c_t}) = \beta(f_1^{c_1} \cdots f_t^{c_t})$ for all $c_1 + \cdots + c_t \leq i$.*

Proof. Considering $\gamma = \alpha - \beta$, it suffices to show $\gamma \in \mathrm{D}^i_{R/A}$ is zero if and only if it is zero on every "monomial"as above.

By induction n i; WLOG $\gamma \in \mathrm{D}^{i-1}$. (Note that $i = 0$ case is clear: $\overline{\gamma} = \overline{s} \Leftrightarrow \gamma = \overline{\gamma}(1) = \overline{s}(1) = s$.) Then, there exists j such that $[\gamma, \overline{f_j}] \neq 0$, since $\gamma \in \mathrm{D}^0_{R/A}$. Otherwise, and by IH, there exist $f_1^{d_1} \cdots f_t^{d_t}$ with $d_1 + \cdots + d_t \leq i-1$, and $[\gamma, \overline{f_j}] f_1^{d_1} \cdots f_t^{d_t} \neq 0$. But, this is $\gamma(f_1^{d_1} \cdots f_j^{d_j+1} \cdots f_t^{d_t}) - f_j(f_1^{d_1} \cdots f_t^{d_t})$, so either $\gamma(f_1^{d_1} \cdots f_j^{d_j+1} \cdots f_t^{d_t})$ or $f_j(f_1^{d_1} \cdots f_t^{d_t})$ is nonzero. □

Theorem 2.0.10. *The two notation of differential operators on $R = K[x_1, \dots, x_n]$, K field of characteristic zero agree. Namely,* $\mathrm{D}_{R/K}$ *by inductive definition equals*

$$K\left\langle \overline{x}_1, \dots, \overline{x}_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle.$$

Proof. Let us write $\widetilde{\mathrm{D}}^i_{R/K}$ for $\bigoplus_{\alpha_1+\cdots+\alpha_n \leq i} \overline{R} \frac{\partial}{\partial x_i}^{\alpha_1} \cdots \frac{\partial}{\partial x_n}^{\alpha_n}$. First, we show $\widetilde{\mathrm{D}}^i_{R/K} \subseteq \mathrm{D}^i_{R/K}$. For $i = 0$, this is clear. Note that

$$\left[\frac{\partial}{\partial x_i}, \overline{x}_j\right] = \begin{cases} \overline{1} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

By Proposition 2.0.6, to check membership in $\mathrm{D}^i_{R/K}$, suffices to check [,] with any generator of R, so $\frac{\partial}{\partial x_j} \in \mathrm{D}^1_{R/K}$ for each j. Then, $\frac{\partial}{\partial x_1}^{\alpha_1} \cdots \frac{\partial}{\partial x_n}^{\alpha_n} \in \mathrm{D}^i_{R/K}$, so $\widetilde{\mathrm{D}}^i_{R/K} \subseteq \mathrm{D}^i_{R/K}$.

For the other containment, we claim that the restriction map $\widetilde{D}^i_{R/K} \longrightarrow \mathrm{Hom}_K(R_{\leq i}, R)$ is an isomorphism. Since $\widetilde{\mathrm{D}}^i_{R/K} \subseteq \mathrm{D}^i_{R/K}$, Proposition 2.0.9 shows that this is injective. To see surjective, we need to see that for any $\{s_\alpha\}_{\alpha_1+\cdots+\alpha_n \leq i} \subseteq R$, there exists $\delta \in \widetilde{\mathrm{D}}^i_{R/K}$ with $\delta(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = s_\alpha$. Recall that

$$\left(\frac{\partial}{\partial x_1}^{\beta_1} \cdots \frac{\partial}{\partial x_n}^{\beta_n}\right)(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = \begin{cases} \beta_1! \cdots \beta_n! & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha_1 + \cdots + \alpha_n \leqslant \beta_1 + \cdots + \beta_n, \\ & \text{and } \alpha \neq \beta \end{cases}.$$

Given $\{s_\alpha\}_{|\alpha| \leq i}$, inductively we can find $\delta' \in \widetilde{\mathrm{D}}^{i-1}_{R/K}$ with $\delta'(\underline{x}^{\underline{\alpha}}) = s_\alpha$ for $|\alpha| \leq i-1$, the take

$$\delta = \delta' + \sum_{|\alpha|=i} \left[\frac{s_\alpha - \delta'(\underline{x}^{\underline{\alpha}})}{\beta_1! \cdots \beta_n!} \left(\frac{\partial}{\partial x_1}^{\alpha_1} \cdots \frac{\partial}{\partial x_n}^{\alpha_n}\right)\right].$$

Now, for any $\delta \in \mathrm{D}^i_{R/K}$, there exists $\widetilde{\delta} \in \widetilde{\mathrm{D}}^i_{R/K}$ such that $\delta|_{R \leq i} = \widetilde{\delta}|_{R \leq i}$. By Proposition 2.0.9 again, we have $\delta = \widetilde{\delta}$. Thus, $\widetilde{\mathrm{D}}^i_{R/K} = \mathrm{D}^i_{R/K}$. □

Corollary 2.0.11. *If K field of characteristic zero, $R = K[x_1, \dots, x_n]$, then the restriction map $\widetilde{D}^i_{R/K} \longrightarrow \operatorname{Hom}_K(R_{\leq i}, R)$ is an isomorphism.*

Exercise 2.0.12. *Let $A \longrightarrow R$ be a map of rings. Then, the associated graded ring of the order filtration of $\mathrm{D}_{R/A}$ (i.e. $gr(\mathrm{D}^{\bullet}_{R/A})$) is commutative.*

2.1 Principal parts

To better understand differential operators, we want an "R-linear" way to compute them, or, to represent them as a functor. To do this, we will need to represent Hom over a subring first.

2.2 Module structures on Hom

Let $A \longrightarrow R$ be a homomorphism of commutative rings, and let M, N be R-modules. There is an A-module homomorphism

$$\begin{aligned} \alpha_M : M \longrightarrow &\quad R \otimes_A M \\ m \longrightarrow &\quad 1 \otimes m. \end{aligned}$$

Observe that there are different R-module structures on $R \otimes_A M$: we can act on R: $r \cdot (s \otimes m) = rs \otimes m$ or act on M: $r \cdot (s \otimes m) = r \otimes sm$, and these differ in general. However, the action of A on R or M agree: $\alpha s \otimes m = s \otimes \alpha m$ for $\alpha \in A$ by bilinearity ver A. there is an $(R \otimes_A R)$-module structure on $R \otimes_A M$: $(a \otimes b)(r \otimes m) = ar \otimes bm$. Note that if $\alpha \in A$, we have $\alpha a \otimes b = a \otimes \alpha b$, $\beta r \otimes m = r \otimes \beta m$, and $\alpha\beta ar \otimes bm = ar \otimes \alpha\beta m$... the action is well-defined.

$\operatorname{Hom}_A(M, N)$ also admits an $(R \otimes_A R)$-module structure. For $\phi \in \operatorname{Hom}_A(M, N)$, $(a \otimes b)\phi = a\phi(bm)$. We have $\alpha a \otimes b = a \otimes \alpha b$, and $\alpha a\phi(bm) = a\phi(\alpha bm)$ by A-linearity.

Theorem 2.2.1. *The map*

$$\begin{aligned} \operatorname{Hom}_R(R \otimes_A M, N) &\xrightarrow{\Phi} \operatorname{Hom}_A(M, N) \\ \psi &\longrightarrow \psi \circ d_M \end{aligned}$$

is an isomorphism of $(R \otimes_A R)$-modules (where the action on LHS is by precomposition).

Proof. First, we verify that this map is $(R \otimes_A R)$-linear: let $\psi \in \operatorname{Hom}_R(R \otimes_A M, N)$. We need to see that

$$\Phi((a \otimes b) \cdot \psi) = (a \otimes b) \cdot \Phi(\psi).$$

Plug in m to both sides:

$$\Phi((a \otimes b) \cdot \Psi)(m) = (a \otimes b)\psi(1 \otimes m) = \psi(a \otimes bm),$$

$$((a\otimes b)\cdot\Phi(\psi))(m)=a(\Phi(\psi)(bm))=a\psi(1\otimes bm).$$

Since ψ is R-linear using R-module structure, we have

$$\psi(a\otimes bm)=\psi(a(1\otimes bm))=a\psi(1\otimes bn).$$

Thus, the module structures are compatibles.

Now we check the map is bijective. By Hom $-\otimes$ adjuntion as A-modules, we have

$$\begin{aligned}\operatorname{Hom}_A(R\otimes_A M,N)\simeq \operatorname{Bil}_A(R\times M,N)&\simeq \operatorname{Hom}_A(R,\operatorname{Hom}_A(M,N))\\ \psi\to\psi\circ((r,m)\to r\otimes m)&\longmapsto (r\to(\psi(r\otimes -)))\end{aligned}$$

We claim that this restricts to the iso we want; i.e, ψ is R-linear if and only if $r\longrightarrow\psi(r\otimes -)\in\operatorname{Hom}_A(R,\operatorname{Hom}_A(M,N))$ is R-linear.

But ψ R-linear $\Leftrightarrow \psi(r\otimes m)=r(1\otimes m)$ for all $r\in R,\ m\in M$, and $r\longrightarrow\psi(r\otimes -)$ R-linear $\Leftrightarrow \psi(r\otimes m)=r\psi(1\otimes m)$. Thus, the iso above restricts to an iso.

$$\begin{aligned}\operatorname{Hom}_R(R\otimes_A M,N)\xrightarrow{\sim}\operatorname{Hom}_R(R,\operatorname{Hom}(M,N))&\xrightarrow{\sim}\operatorname{Hom}_A(M,N)\\ \psi\longrightarrow(r\longrightarrow\psi(r\otimes -))&\longrightarrow\psi(1\otimes -),\end{aligned}$$

and this is just the map Φ. □

We will use these module structures to characterize differential operators. It begins with an observation:

Lemma 2.2.2. *Let $\alpha\in\operatorname{Hom}_A(M,N)$ and $f\in R$. Then, $[\alpha,\overline{f}]=(1\otimes f-f\otimes 1)\cdot\alpha$ under the $R\otimes_A R$-module structure specified.*

Proof. We have $(1\otimes f)\cdot\alpha(-)=\alpha(f-)$, so $(1\otimes f)\cdot\alpha=\alpha\overline{f}$ and $(f\otimes 1)\cdot\alpha(-)=f\alpha(-)$, so $(f\otimes 1)\alpha=\overline{f}\alpha$. □

Let $A\longrightarrow R$ commutative rings. Then, there is a ring homomorphism $R\otimes_A R\xrightarrow{\mu}R$ given by $\mu(r\otimes s)=rs$. We defines

$$\Delta_{R/A}=\operatorname{Ker}(R\otimes_A R\xrightarrow{\mu}R).$$

We observe that $\Delta_{R/A}$ is generated by elements of the form $1\otimes f-f\otimes 1$, $f\in R$. Each such element is in the kernel, and there is an isomorphism $(R\otimes_A R)/(\{1\otimes f-f\otimes 1\})\simeq R$, since every element of $R\otimes_A R$ is equivalent modulo $(\{1\otimes f-f\otimes 1\})$ to an element of the $r\otimes 1$, and there is an R-linear inverse $r\longrightarrow r\otimes 1+(\{1\otimes f-f\otimes 1\})$.

Moreover, if $R=A[f_1.\dots,f_t]$, then $\Delta_{R/A}=(1\otimes f_1-f_1\otimes 1,\cdots,1\otimes f_t-f_t\otimes 1)$. This follows from the identity

$$1\otimes fg-fg\otimes 1=(1\otimes f-f\otimes 1)(1\otimes g)+(1\otimes g-g\otimes 1)(f\otimes 1)$$

Proposition 2.2.3. *The collection* $\mathrm{D}^i_{R/A}(M,N) \subseteq \mathrm{Hom}_A(M,N)$ *is the* $R \otimes_A R$*-submodule of* $\mathrm{Hom}_A(M,N)$ *annihilated by* $\Delta^{i+1}_{R/A}$.

Proof. By induction on i. For $i = 0$, $\delta \in \mathrm{D}^0_{R/A}(M,N)$ if and only if $[\delta, overliner] = 0$ for each $r \in R$, i.e., iff each $[\,\cdot\,, \overline{r}]$ operation annihilates it. This is equivalent to each element $(1 \otimes r - r \otimes 1)$ annihilating δ, which is equivalent to $\Delta_{R/A}$ annihilating δ.

The inductive step is similar: for the same reason, we have

$$\begin{aligned} \delta \in \mathrm{D}_{R/A}(M,N) &\Leftrightarrow \Delta_{R/A} \cdot \delta \in \mathrm{D}^{i-1}_{R/A}(M,N) \\ &\Leftrightarrow \Delta_{R/A} \cdot \Delta^i_{R/A} \cdot \delta = 0 \textbf{ for IH} \\ &\Leftrightarrow \Delta^{i+1}_{R/A} \cdot \delta = 0. \end{aligned}$$

□

Definition 2.2.4. Set $\mathrm{P}^i_{R/A} := (R \otimes_A R)/\Delta^{i+1}_{R/A}$ and $\mathrm{P}^i_{R/A}(M) := (R \otimes_A M)/\left(\Delta^{i+1}_{R/A}(R \otimes_A M)\right)$. These are $(R \otimes_A R)$-modules, and we view them as R-modules by the action on the left coy of $R \otimes_A R$ or $R \otimes_A M$. I.e.,

$$r \cdot (a \otimes m + \Delta^{i+1}_{R/A}(R \otimes_A M)) = ra \otimes m + \Delta^{i+1}_{R/A}(R \otimes_A M).$$

We call $\mathrm{P}^i_{R/A}$ the **module of principal parts** for every i.

Theorem 2.2.5. *There is an isomorphism of* $R \otimes_A R$*-modules.*

$$\mathrm{Hom}_R(\mathrm{P}^i_{R/A}(M), N) \xrightarrow{\sim} \mathrm{D}^i_{R/A}(M,N).$$

This isomorphism is given by the composition

$$\mathrm{Hom}_R(\mathrm{P}^i_{R/A}(M), N) \xrightarrow{\pi^*} \mathrm{Hom}_R(R \otimes_A M, N) \xrightarrow{d^*_M} \mathrm{D}^i_{R/A}(M,N),$$

where π^* *is the map* $\mathrm{Hom}_(\pi, N)$ *induced by the quotient* $R \otimes_A M \overset{\pi}{\twoheadrightarrow} \mathrm{P}^i_{R/A}(M)$*, and* d^*_M *is the (restriction) map* Φ *that is precomposition by* d_M*.*

Proof. Consider the SES of $R \otimes R$-modules or R-modules

$$0 \longrightarrow \Delta^{i+1}_{R/A}(R \otimes M) \longrightarrow R \otimes M \xrightarrow{\pi} \mathrm{P}^i(M) \longrightarrow 0.$$

By left exactness of $\mathrm{Hom}_R(,N)$, we have

$$0 \longrightarrow \mathrm{Hom}_R(\mathrm{P}^i(M), N) \xrightarrow{\pi^*} \mathrm{Hom}_R(R \otimes_A M, N) \longrightarrow \mathrm{Hom}_R(\Delta^{i+1}(R \otimes M), N) \longrightarrow 0$$

exact, which means π^* is injective, and its image is the ser of maps that are zero on $\Delta^{i+1}(R \otimes M)$. Since we view $\mathrm{Hom}_R(R \otimes_A M, N)$ as a module via the action on the source, φ is zero on $\Delta^{i+1}(R \otimes M) \Leftrightarrow \Delta^{i+1} \cdot \varphi = 0$ in $\mathrm{Hom}(R \otimes M, N)$. Thus, $d_M \circ \pi^*$ induces an isomorphism between $\mathrm{Hom}_R(\mathrm{P}^i(M), N)$ and $\mathrm{Ann}_{\mathrm{Hom}_A(M,N)}(\Delta^{i+1}_{R/A}) = \mathrm{D}^i_{R/A}$. □

Example 2.2.6. Let A be a ring, and $R = A[x_1, \ldots, x_n]$ polynomial ring over A. We can write $R \otimes_A R \simeq A[x_1, \ldots, x_n, y_1, \ldots, y_n]$ $(x_i = x_i \otimes 1,\ y_i = y_i \otimes 1)$. We view $R \otimes_A R$ as an R-algebra by the left inclusion map and we identify elements with their image in an algebra like usual. Then, $d_R(x_i) = y_i$ for each i, so $d_R(f(\underline{x})) = f(\underline{y})$.

The ideal $\Delta_{R/A} = (y_1 - x_1, \ldots, y_n - x_n)$. Let us rewrite $z_i = y_i - x_i$, so $R \otimes_A R \simeq A[x_1, \ldots, x_n, z_1, \ldots, z_n]$ and $\Delta_{R/A} = (z_1, \ldots, z_n)$. Now, $d_R(x_i) = x_i + z_i$, so $d_R(f(\underline{x})) = f(\underline{x} + \underline{z})$. We compute

$$\mathrm{P}^i_{R/A} \simeq A[\underline{x}, \underline{z}]/(\underline{z})^{i+1} \simeq \bigoplus_{\alpha_1 + \cdots + \alpha_n \leq i} R z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

as R-modules. This is a free module on the generators $\{z^\alpha \mid |\alpha| \leq i\}$. Let $\{(z^\alpha)* \mid |\alpha| \leq i\}$ be the dual basis: i.e., $(z^\alpha)*$ returns the z^α-component in the unique expression of an element as a sum as above. Thus, $\mathrm{Hom}_R(\mathrm{P}^i_{R/A}, R) \simeq \bigoplus_{|\alpha| \leq i} R(z^\alpha)^*$.

We now want to identify $(z^\alpha)^* \circ d_R$ as differential operators.

Definition 2.2.7. Let $R = A[\underline{x}]$, and $\partial^{(\alpha)} : R \longrightarrow R$ be the map A-linear on R defined by

$$\partial^{(\alpha)}(x_1^{\beta_1} \cdots x_n^{\beta_n}) = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n} x_1^{\beta_1 - \alpha_1} \cdots x_n^{\beta_n - \alpha_n}.$$

If $\mathbb{Q} \subseteq A$, we can identify $\partial^{(\alpha)}$ with $\frac{\partial}{\partial x_1}^{\alpha_1} \cdots \frac{\partial}{\partial x_n}^{\alpha_n} \cdot \frac{1}{\alpha_1! \cdots \alpha_n!}$ as $\binom{\beta_i}{\alpha_i} = \frac{\beta_i(\beta_i - 1) \cdots (\beta_i - \alpha_i + 1)}{\alpha_i(\alpha_i - 1) \cdots 1}$. In general, $1/(\alpha_1! \cdots \alpha_n!)$ is necessarily meaningful in A.

Lemma 2.2.8 (Taylor expansions)**.** *In $R = A[\underline{x}]$, we have $f(\underline{x} + \underline{z}) = \sum_a \alpha(\partial^{(\alpha)} f)(\underline{x}) \cdot z^\alpha$.*

Proof. It suffices to cheek for monomials, in which case this is basically the binomial coefficient. □

Theorem 2.2.9. *Let $R = A[x_1, \ldots, x_n]$ be a polynomial ring over another ring. Then, $\mathrm{D}^i_{R/A} = \bigoplus_{|\alpha| \leq i} \overline{R} \partial^{(\alpha)}$.*

Proof. We computed that $\mathrm{Hom}_R(\mathrm{P}^i_{R/A}, R)$ is freely generated by $(z^\alpha)^*$. From Lemma 2.2.8, $\mathrm{D}^i_{R/A}$ is freely generated by $\partial^{(\alpha)}$. □

Exercise 2.2.10. *Show that $\mathrm{D}^i_{A[x_1, \ldots, x_n]/A}$ is freely generated A-algebra if and only if $\mathbb{Q} \subseteq A$.*

We also note the following:

Proposition 2.2.11. *Let $R = A[\underline{x}]$ be a polynomial ring. Then, $\mathrm{P}^i_{R/A}$ is a free R-module.*

Our next goal is to give some sort of description of differential operators on finitely generated algebras. The key is the following:

Proposition 2.2.12. *Let $S = A[x_1, \dots, x_n]$ be a polynomial ring, $I \subseteq S$ be an ideal, and $R = S/I$. Then, for every $\delta \in \mathrm{D}^i_R$ there exists $\widetilde{\delta} \in \mathrm{D}^i_S$ such that the diagram*

$$\begin{array}{ccc} S & \overset{\widetilde{\delta}}{\dashrightarrow} & S \\ \downarrow & & \downarrow \\ R & \xrightarrow{\delta} & R \end{array}$$

commutes.

Proof. Note that $R \otimes_A R \simeq S/I \otimes S/I$ is quotient of $S \otimes_A S$, and $\Delta_{S/A}$ maps to $\Delta_{R/A}$ under this quotient. Thus, we have a surjection $\mathrm{P}^i_{S/A} \twoheadrightarrow \mathrm{P}^i_{R/A}$; moreover the diagram

$$\begin{array}{ccc} S & \xrightarrow{d_S} & \mathrm{P}^i_{S/A} \\ \downarrow & & \downarrow \\ R & \xrightarrow{d_R} & \mathrm{P}^i_{R/A} \end{array}$$

commutes.

Given δ write $\delta = \varphi \circ d_R$, so we have

$$\begin{array}{ccccc} S & \xrightarrow{d_S} & \mathrm{P}^i_{S/A} & \overset{\widetilde{\varphi}}{\dashrightarrow} & S \\ \downarrow & & \downarrow & & \downarrow \\ R & \xrightarrow{d_R} & \mathrm{P}^i_{R/A} & \xrightarrow{\varphi} & R. \end{array}$$

Then, $\mathrm{P}^i_{S/A} \longrightarrow R$ is S-linear, and by freeness ($\mathrm{P}^i_{S/A}$ is projective), there is (S-linear a lift $\widetilde{\varphi}$ making the diagram commute. Then, $\widetilde{\varphi} \circ d_S = \widetilde{\delta}$ works. □

Theorem 2.2.13. *Let $S = A[x_1, \dots, x_n]$ be a polynomial ring, $I \subseteq S$ be an ideal, and $R = S/I$. Then,*

$$\mathrm{D}^i_{R/A} \simeq \frac{\{\delta \in \mathrm{D}^i_{S/A} \mid \delta(I) \subseteq I\}}{I\,\mathrm{D}^i_{S/A}}.$$

Proof. By Proposition 2.2.12 every operator in $\mathrm{D}^i_{R/A}$ lifts to an operator in $\mathrm{D}^i_{S/A}$. An operator $\delta \in \mathrm{D}^i_{S/A}$ is the lift of some element in $\mathrm{D}^i_{R/A}$ if and only if $\delta(I) \subseteq I$ (iff $\delta(S + I) \subseteq \delta(S) + I$). Thus,

$$\mathrm{D}^i_{R/A} \simeq \frac{\{\delta \in \mathrm{D}^i_{S/A} \mid \delta(I) \subseteq I\}}{\{\delta \in \mathrm{D}^i_{S/A} \mid \delta(S) \subseteq I\}}.$$

Exercise to check $\delta(S) \subseteq I$ iff $\delta \in I\,\mathrm{D}^i_{S/A}$. □

Now, we use principal parts to investigate the behavior of differential operators under localization.

Proposition 2.2.14. *Let $A \longrightarrow R$ commutative, and $W \subseteq R$ multiplicative closed. Then,*

$$W^{-1}\,\mathrm{P}^i_{R/A} \simeq \mathrm{P}_{W^{-1}R/A} \simeq \mathrm{P}^i_{W^{-1}R/(W\cap A)^{-1}A}\,.$$

Proof.

$$W^{-1}\,\mathrm{P}^i_{R/A} = (W\otimes 1)^{-1}\left(\frac{R\otimes_A R}{\Delta^{i+1}_{R/A}}\right),$$

and

$$\mathrm{P}_{W^{-1}R/A} = \left(\frac{W^{-1}R\otimes_A W^{-1}R}{\Delta^{i+1}_{W^{-1}R/A}}\right) \simeq (W\otimes 1)^{-1}(1\otimes W)^{-1}\left(\frac{R\otimes_A R}{\Delta^{i+1}_{R/A}}\right),$$

so we need to show that any element of the form $1\otimes w$, $w\in W$ is a unit in $(W\otimes 1)^{-1}\left(\frac{R\otimes_A R}{\Delta^{i+1}_{R/A}}\right)$. But, $1\otimes w = w\otimes 1 + (1\otimes w - w\otimes 1)$, a unit plus a nilpotent, which is a unit (since for any ideal I, nilpotent in I, unit is not in I). This justifies teh first isomorphism. For the second, the map

$$W^{-1}R\otimes_A W^{-1}R \longrightarrow W^{-1}R\otimes_{(W\cap A)^{-1}A} W^{-1}R$$

sending $a/w\otimes b/v \to a/w\otimes b/v$ is well-defined (since there is an A-bilinear map $W^{-1}R\times_A W^{-1}R \longrightarrow W^{-1}R\otimes_{(W\cap A)^{-1}A} W^{-1}R$), and the inverse is well-defined too, since

$$\frac{a}{w}\frac{\alpha}{\mu}\otimes\frac{b}{w} = \frac{a(\alpha\mu)}{\mu w}\otimes\frac{b}{\mu v} = \frac{a}{\mu w}\otimes\frac{(\alpha\mu)b}{\mu v} = \frac{a}{w}\otimes\frac{\alpha}{\mu}\frac{b}{v}$$

in $W^{-1}R\otimes_A W^{-1}R$. This induces the isomorphism on the respective quotient. □

Proposition 2.2.15. *Let R be a localization of a finitely generated A-algebra. Then, $\mathrm{P}^i_{R/A}$ is a finitely generated R-module.*

Proof. By Proposition 2.2.14, it suffices to deal with the case R is finitely generated. Write $R = S/I$, for S a finitely generated polynomial ring over A. We then have

$$\mathrm{P}^i_{S/A} = \frac{S\otimes_A S}{\Delta^{i+1}_{S/A}} \twoheadrightarrow \frac{S\otimes_A S}{I\otimes 1 + 1\otimes I + \Delta^{i+1}_{S/A}} \simeq \frac{R\otimes_A S}{1\otimes I + \Delta^{i+1}_{S/A}(R\otimes_A S)} \simeq \mathrm{P}^i_{R/A}\,.$$

A generating set for $\mathrm{P}^i_{S/A}$ serves as a generated set for $(R\otimes_A S)/\Delta^{i+1}_{S/A}(R\otimes_A S)$, which surject into $\mathrm{P}^i_{R/A}$. The claim follows. □

Theorem 2.2.16. *Let A be a Noetherian ring, R be a localization of a finitely generated A-algebra, and M be an R-module. Then,*

$$\mathrm{D}^i_{W^{-1}R/A}(W^{-1}R, W^{-1}M) \simeq \mathrm{D}^i_{W^{-1}R/(W\cap A)^{-1}A}(W^{-1}R, W^{-1}M) \simeq W^{-1}\,\mathrm{D}^i_{R/A}(R,M).$$

Proof. We have

$$\begin{aligned}\mathrm{D}^i_{W^{-1}R/A}(W^{-1}R, W^{-1}M) &\simeq \mathrm{Hom}_{W^{-1}R}(\mathrm{P}^i_{W^{-1}R/A}, W^{-1}M)\\ &\simeq \mathrm{Hom}_{W^{-1}R}(W^{-1}\,\mathrm{P}^i_{R/A}, W^{-1}M)\\ &\simeq W^{-1}\,\mathrm{Hom}_R(\mathrm{P}^i_{R/A}, M)\end{aligned}$$

by hypothesis, since $\mathrm{P}^i_{R/A}$ is finitely generated, and R is Noetherian, then $\mathrm{P}^i_{R/A}$ has finite presentation. Thus, $\mathrm{D}^i_{W^{-1}R/A}(W^{-1}R, W^{-1}M) \simeq W^{-1}\,\mathrm{D}^i_{R/A}(R,M)$.

More generally, this holds whenever $\mathrm{P}^i_{R/A}$ is finitely generated.

The other is similar. □

This theorem says that in this case every differential operator on $W^{-1}R$ is $w \circ \widetilde{\delta}$ for $\widetilde{\delta}$ an extension of an operator on R to $W^{-1}R$. We can make this more explicit.

Let $\delta \in \mathrm{D}^i_{R/A}(R,M)$ and $w \in W$. If $\widetilde{\delta}$ is an extension of δ to $\mathrm{D}^i_{W^{-1}R/A}(W^{-1}R, W^{-1}M)$ we must have

$$\begin{aligned}\delta(r) &= \delta\left(w\frac{r}{w}\right)\\ &= (\widetilde{\delta}\overline{w})\left(\frac{r}{w}\right)\\ &= (\overline{w}\widetilde{\delta} + [\widetilde{\delta}, \overline{w}])\left(\frac{r}{w}\right)\\ &= w\widetilde{\delta}\left(\frac{r}{w}\right) + [\widetilde{\delta}, \overline{w}]\left(\frac{r}{w}\right),\end{aligned}$$

so

$$\widetilde{\delta}\left(\frac{r}{w}\right) = \frac{\delta(r) + [\widetilde{\delta}, \overline{w}])(\frac{r}{w})}{w}$$

and $[\widetilde{\delta}, \overline{w}]$ has smaller order, so this inductively gives an operator. Respecting, write $\delta^{(0)} = \delta$,

$$\delta^{(j)} = [\delta^{(j-1)}, \overline{w}] \in \mathrm{D}^{i-j}_{R/A}(R,M);$$

we get

$$\widetilde{\delta}(r/w) = \sum_{j=0}^{i} \frac{\delta^{(j)}(r)}{w^{j+1}}.$$

Chapter 3

Zariski-Nagata theorems on symbolic powers

Our next application is to symbolic powers. We want to prove a theorem of Zariski-Nagata and more general version by Cid-Ruiz building on work of Blumfiel.

We will use colon notation

$$(N :_R T) = \{r \in R \mid t(r) \in N \; \forall t \in T\}$$

for a collection of differential operators T from $R \longrightarrow M$ and some submodule N of M.

Our goal is the following theorem:

Theorem 3.0.1. *Let K be a perfect field, R be an algebra essential of finite type over K, and $P \subseteq R$ prime. Then,*

1) $P^{(n)} = (0 :_R \mathrm{D}^{n-1}_{R/K}(R, R_P))$.

2) *If R (localization of) is a polynomial ring, then* $P^{(n)} = (P :_R \mathrm{D}^{n-1}_{R/K})$.

First containment. This will require a bit of preparation. First we study the ideals specified on the RHS.

Proposition 3.0.2. *For each n, and any ideal I, $(0 :_R \mathrm{D}^n_{R/A}(R, R/I))$ and $(I :_R \mathrm{D}^n_{R/A})$ are ideals.*

Proof. By induction on n. For $n = 0$, $(0 :_R \mathrm{D}^0_{R/A}(R, R/I)) = (I :_R \mathrm{D}^0_{R/A}) = I$.

Inductive step: Additivity is clear since operators are additive. If $r \in R$ with $\mathrm{D}^n_{R/A}(R, R/I) \cdot r = 0$ and $s \in R$ general, $\mathrm{D}^n_{R/A}(R, R/I) \circ \overline{s} \subseteq \mathrm{D}^n_{R/A}(R, R/A)$, implies ok. Likewise for $\mathrm{D}^n_{R/A}(R, R/I) \cdot r \in I$. □

Proposition 3.0.3. *We have $I(0 :_R \mathrm{D}^i_{R/A}(R, R/I)) \subset (0 :_R \mathrm{D}^{i+1}_{R/A}(R, R/I))$ and $I(I :_R \mathrm{D}^i_{R/A}) \subset (I :_R \mathrm{D}^{i+1}_{R/A})$.*

Proof. Let $\mathrm{D}^i_{R/A}(R, R/I) \cdot r = 0$ and $f \in I$. If $\delta \in \mathrm{D}^{i+1}_{R/A}(R, R/I)$, then

$$\delta(fr) = \delta\overline{f}(r) = \overline{f}\delta(r) + [\delta, f](r) = f\delta(r) + 0 = 0.$$

Similarly for other. □

Proposition 3.0.4. *If P is prime, then $(0 :_R \mathrm{D}^i_{R/A}(R, R/P))$ and $(P :_R \mathrm{D}^i_{R/A})$ are P-primary.*

Proof. Induction we show $r \notin P$ $ra \in I \Rightarrow a \in I$ for these ideals. $\mathrm{D}^{i+1}_{R/A}(R, R/P) \cdot (ra) = 0$, $r \notin P$, take $\delta \in \mathrm{D}^{i+1}_{R/A}(R, R/P)$. Then

$$0 = \delta(ra) = (\delta\overline{r})(a) = (\overline{r}\delta)(a) + [\delta, \overline{r}](a) = r\delta(a) + [\delta, \overline{r}](a).$$

By IH, $[\delta, \overline{r}](a) \subseteq \mathrm{D}^i_{R/A}(R, R/P) \cdot a = 0$, and $r\delta(a) = 0$, then $\delta(a) = 0$.
Similar for other. □

This gives the containment $P^{(n)} \subseteq (0 :_R \mathrm{D}^{n-1}_{R/A}(R, R/P))$, since $P^n \subseteq (0 :_R \mathrm{D}^{n-1}_{R/A}(R, R/P))$ by above, and $P^{(n)}$ is smallest P-primary ideal containing P^n.

Proposition 3.0.5. *If R is e.f.t/K, $P \subseteq R$ prime. Then,*

$$(0 :_R \mathrm{D}^{n-1}_{R/K}(R, R/P))_P = (0 :_{R_P} \mathrm{D}^{n-1}_{R/K}(R_P, R_P/PR_P)).$$

Proof. Can write an element of LHS a r/w with $\mathrm{D}^{n-1}_{R/K}(R, R/P) \cdot r = 0$, $w \notin P$.

If $\delta \in \mathrm{D}^{n-1}_{R/K}(R_P, R_P/PR_P)$, write $\delta = \frac{1}{v}\alpha$, $v \notin P$, and α extension of an operator in $\mathrm{D}^{n-1}_{R/K}(R, R/P)$. Then, $\delta(r/w) = \frac{1}{vw}\left(\alpha(r) + \frac{\alpha^{(1)}(r)}{w} + \frac{\alpha^{(2)}(r)}{w^2} + \cdots + \frac{\alpha^{(n-1)}(r)}{w^{n-1}}\right) = 0$ ($\alpha^{(i)} \in \mathrm{D}^{n-1-i}_{R/K}(R, R/P)$), so ($\subseteq$) holds.

If $\mathrm{D}^{n-1}_{R/K}(R_P, R_P/PR_P) \cdot \frac{r}{w} = 0$ ($w \notin P$), $\mathrm{D}^{n-1}_{R/K}(R_P, R_P/PR_P) \cdot r = 0$, and $\mathrm{D}^{n-1}_{R/K}(R, R/P) \cdot r = 0$ (since every operator extend to the former type), so ($\supseteq$) too. □

Thus, it only remains for (i) to show that if $(R, \mathfrak{m}, K)$ is local, e.f.t over K perfect, then $(0 :_R \mathrm{D}^{n-1}_{R/K}(R, K)) = \mathfrak{m}^n$.

3.1 Separable field extensions and quasicoefficient field

Theorem 3.1.1 (Maclane)**.** *Let $K \subseteq L$ be a finitely generated extension of fields, with K perfect. Then, there exist $x_1, \ldots, x_t \in L$ such that $K \subseteq K(x_1, \ldots, x_t)$ purely transcendental and $K(x_1, \ldots, x_t) \subseteq L$ finite separable.*

Proof. If $\operatorname{char}(K) = 0$ no problem: any transcendence basis works. (Note that $L/K(x_1, \ldots, x_t)$ algebraic implies finite, since L/K is finitely generated.) In characteristic $p > 0$, let $F^{sep} = \{\ell \in L \mid \ell \text{ separable algebraic over } F\}$ for $F \subseteq L$ subfield. Pick $x_1, \ldots, x_t \in L$ so that $[L : K(x_1, \ldots, x_t)^{sep}]$ is minimal among all transcendence basis $\{x_1, \ldots, x_t\}$ (again this is finite).

Suppose there exists $y \in L$ inseparable over $K(x_1, \ldots, x_t)$, let $F(z)$ be a its minimal polynomial (note all exponents are multiples of p). Write

$$F(z) = \frac{f_0(\underline{x})}{g_0(\underline{x})} z^n + \frac{f_1(\underline{x})}{g_1(\underline{x})} z^{n-1} + \cdots + \frac{f_n(\underline{x})}{g_n(\underline{x})}$$

where $f_i, g_i \in K[\underline{x}]$ and f_i, g_i are coprime for each i. Clear denominators (multiply by $g_0 \cdots g_n$) we get a polynomial $H(z) \in K[\underline{x}][z]$ with $H(y) = 0$. The coefficients of $H(z)$ are now coprime, and $H(z) = \frac{1}{g_0 \cdots g_n} F(z)$ is irreducible in $K(\underline{x})[z]$, so by Gauss' Lemma, $H(z)$ is irreducible.

Now, not every exponent of each x_j is a multiple of p, else H is a p^{th} power (since K perfect), so WLOG x_n o cuss somewhere without a multiple of p exponent. Thus, x_n is separable algebraic over $K(x_1, \ldots, x_{n-1}, y)$, and $x_1, \ldots, x_{n-1}, y$ is a transcendence basis for L/K. The, $x_n, y \in K(x_1, \ldots, x_{n-1}, y)^{sep} \supsetneq K(x_1, \ldots, x_n)^{sep}$, contradicting the choice of $x_1, \ldots, x_n$. □

Definition 3.1.2. If $(R, \mathfrak{m}, k)$ is a local ring, we say L is a quasicoefficient field for R if $L \subseteq R$ and $L \subseteq R \twoheadrightarrow k$ is finite separable.

Proposition 3.1.3. *If K is perfect, and $(R, \mathfrak{m}, k)$ is an eft/K, then there exists a quasicoefficient field L for R containing K.*

Proof. $K \subseteq k$ is a finitely generated field extension: it is generated by the image of the gens of R up to localization. Pick $x_1, \ldots, x_t \in R$ such that the images $\{\widetilde{x}_i\}$ in $R/\mathfrak{m} \simeq k$ are algebraly independent and $K(\widetilde{x}_1, \ldots, \widetilde{x}_t) \subseteq k$ is finitely separable, by Theorem 3.1.1. We claim that $\{x_1, \ldots, x_t\}$ are algebraly independent over K; otherwise, an algebraic relation on these would give a nonzero algebraic relation the $\widetilde{x}_i'^s$ in k.

We the observe that any nonzero polynomial in the $x_i'^s$ is a unit in R; else we get a relation modulo $\mathfrak{m}$ (i.e. on the $\widetilde{x}_i'^s$ in k). Thus, $K(x_1, \ldots, x_t) = L$ is a purely transcendental over K subfield of R, and the image of L to k is finite separable. That is, L is a quasicoefficient field. □

Example 3.1.4. $\mathbb{R}[x]_{(x^2+1)}$ has no coefficient field, since there is no solution to $f^2+1 = 0$ in $\mathbb{R}(x) \supseteq \mathbb{R}[x]_{(x^2+1)}$. $\mathbb{R}$ is a quasicoefficient field though.

Example 3.1.5. $\mathbb{F}_p(t)[x]_{(x^p-t)}$ has no quasicoefficent field, even, since thre is no solution to $f^p = t$ in $\mathbb{F}_p(t, x) \supseteq \mathbb{F}_p(t)[x]_{(x^p-t)}$.

Proposition 3.1.6. *Let $(R, \mathfrak{m}, k)$ is a local ring with quasicoefficient field L. Then, $k \otimes_R \mathrm{P}^n_{R/L} \simeq R/\mathfrak{m}^{n+1}$ as $R \otimes_L R$-modules, where the left action an RHS is described below, and right action is the usual R-action.*

Proof. We have $k \otimes_R \mathrm{P}^n_{R/L} \simeq k \otimes \frac{R \otimes_L R}{\Delta^{n+1}_{R/L}} \simeq \frac{k \otimes_L R}{\overline{\Delta}^{n+1}_{R/L}}$, where $\overline{\Delta}_{R/L}$ is the image of $\Delta_{R/L}$ modulo $\mathfrak{m} \otimes_L R$.

We can write $k = L(\lambda) = L[T]/(f(T))$ by primitive element theorem, where f is the minimal polynomial of λ over L. Then, $f'(\lambda)$ is nonzero in k by separability. If $\gamma \in R$ has image to $\lambda \in R/\mathfrak{m} = k$, then $f(\gamma) \in \mathfrak{m}$ and $f'(\gamma)$ is a unit in R. By Hensel's Lemma, we can pick γ with $f(\gamma) \in \mathfrak{m}^{n+1}$.

We claim that $\overline{\Delta}_{R/L} = 1 \otimes \mathfrak{m} + (1 \otimes \gamma - \lambda \otimes 1)$. Indeed, we can write $r = m + g(\gamma)$ for $m \in \mathfrak{m}$, $g(T) \in L[T]$, and

$$\begin{aligned} 1 \otimes r - r \otimes 1 &= (1 \otimes m - m \otimes 1) + (1 \otimes g(\gamma) - g(\gamma) \otimes 1) \\ &= 1 \otimes m + (1 \otimes g(\gamma) - g(\lambda) \otimes 1) \end{aligned}$$

modulo $\mathfrak{m} \otimes_L R$, and the latter is an $R \otimes_L R$-linear combination of $1 \otimes \gamma - \lambda \otimes 1$ (exercise).

Now, $k \otimes_L R \simeq R[x]/(f(x))$ as L-algebras. "Left R" acts by taking $r \pmod{\mathfrak{m}} \in k$ as a polynomial in $L[x]/(f(x))$, and multiplying "Right R" is the usual action.

The image of $\overline{\Delta}_{R/L}$ is $\mathfrak{m} + (\gamma - x) := Q$.

We claim that $\mathfrak{m}^{n+1} + (\gamma - x) = Q^{n+1}$ in $R[x]/(f(x))$. "$(\supseteq)$" is clear, to see equality, note that Q is the only maximal ideal containing $\mathfrak{m}^{n+1} + (\gamma - x)$, so we may localize at Q and apply NAK. But we have

$$0 = f(x) = f((x - \gamma) + \gamma) = f(\gamma) + f'(\gamma)(x - \gamma) + H(x - \gamma)^2.$$

Thus, $x - \gamma = \mathfrak{m} + (x - \gamma)^2 \subseteq \mathfrak{m}^{n+1} + (x - \gamma)^2$. Then,

$$\mathfrak{m}^{n+1} + (\gamma - x) \subseteq Q^{n+1} + Q(\mathfrak{m}^{n+1} + (\gamma - x)$$

holds, since $\mathfrak{m}^{n+1} \subseteq Q^{n+1}$ and $(\gamma - x) \subseteq Q^{n+1} + (\gamma - x)Q$.

Thus,

$$(R[x]/(f(x)))/Q^{n+1} \simeq (R[x]/(f(x)))/(\mathfrak{m}^{n+1} + (\gamma - x)) \simeq R/\mathfrak{m}^{n+1},$$

since $f(\gamma) \in \mathfrak{m}^{n+1}$, as required. The left action of R is by valuating modulo $\mathfrak{m}$ and writing as $g(\lambda)$, then acting by $g(\gamma)$. □

Proposition 3.1.7. *Let $(R, \mathfrak{m}, k)$ be local with quasicoefficient field L. Then, $\mathrm{D}^n_{R/L}(R, k) \simeq \mathrm{Hom}_k(R/\mathfrak{m}^{n+1}, k)$ (by the left R-module structure above). In particular, if $f \notin \mathfrak{m}^{n+1}$, there exists $\delta \in \mathrm{D}^n_{R/L}(R, k)$ such that $\delta(f) = 1$.*

Proof. We have

$$\mathrm{D}^n_{R/L}(R,k) \simeq \mathrm{Hom}_R(\mathrm{P}^n_{R/L},k) \simeq \mathrm{Hom}_k(k \otimes_R \mathrm{P}^n_{R/L},k) \simeq \mathrm{Hom}_k(R/\mathfrak{m}^{n+1},k).$$

The second claim comes from observing that a nonzero element in $R/\mathfrak{m}^{n+1}$ is part of a k-basis. □

Theorem 3.1.8. *Let K be a perfect field and R be essential of finite over K, $P \subseteq R$ prime. Then, $P^{(n)} = (0 :_R \mathrm{D}^{n-1}_{R/K}(R,R/P))$.*

Proof. We already have $P^{(n)} \subseteq (0 :_R \mathrm{D}^{n-1}_{R/K}(R,R/P))$. It suffices to show $PR_P = P^{(n)}R_P \supseteq (0 :_R \mathrm{D}^{n-1}_{R/K}(R,R/P))R_P$. We have RHS$= (0 :_{R_P} \mathrm{D}^{n-1}_{R_P/K}(R_P,R_P/PR_P))$.

Write $(R_P,PR_P,R_P/PR_P) = (S,\mathfrak{n},k)$: S is essential of finite type over perfect K. There is a quasicoefficient field L for S. Now, $(0 :_S \mathrm{D}^{n-1}_{S/K}(S,k)) \subseteq (0 :_S \mathrm{D}^{n-1}_{S/L}(S,k))$, since $K \subseteq L$, $\mathrm{D}^{n-1}_{S/L}(S,k) \subseteq \mathrm{D}^{n-1}_{S/K}(S,k)$. But, if $f \notin \mathfrak{n}^n$, then the image of f is nonzero in $S/\mathfrak{n}^n$, so there is a map in $\mathrm{Hom}_L(S/\mathfrak{n}^n,k) \simeq \mathrm{D}^{n-1}_{S/L}(S,k)$ taking f to something nonzero, so $f \notin (0 :_S \mathrm{D}^{n-1}_{S/L}(S,k))$, as required. □

Theorem 3.1.9. *Let K be a perfect field, and R be (a localization) a polynomial ring over K, $P \subseteq R$ prime. Then, $P^{(n)} = (P :_R \mathrm{D}^{n-1}_{R/K})$.*

Proof. We just need to show that $(P :_R \mathrm{D}^{n-1}_{R/K}) = (0 :_R \mathrm{D}^{n-1}_{R/K}(R,R/P))$ in this case. From the short exact sequence

$$0 \longrightarrow P \longrightarrow R \longrightarrow R/P \longrightarrow 0,$$

we get

$$0 \longrightarrow \mathrm{Hom}_R(\mathrm{P}^n_{R/K},P) \longrightarrow \mathrm{Hom}_R(\mathrm{P}^n_{R/K},R) \longrightarrow \mathrm{Hom}_R(\mathrm{P}^n_{R/K},R/P) \longrightarrow 0.$$

Since $\mathrm{P}^n_{R/K}$ is a free module (crucial). Thus,

$$\begin{aligned} \mathrm{D}^n_{R/K} &\longrightarrow \mathrm{D}^n_{R/K}(R,R/P) \\ \delta &\longrightarrow \pi \circ \delta, \end{aligned}$$

where $\pi : R \longrightarrow R/P$ projection is surjective. That is, every operator to R/P comes from operator $R \longrightarrow R$. Thus, if $r \in (P :_R \mathrm{D}^{n-1}_{R/K})$, and $\beta \in \mathrm{D}^{n-1}_{R/K}(R,R/P)$, write $\beta = \pi\circ\delta$, $\delta \in \mathrm{D}^{n-1}_{R/K}$, so $\beta(r) = \pi(\delta(r)) = 0$. Thus, $(P :_R \mathrm{D}^{n-1}_{R/K}) \subseteq (0 :_R \mathrm{D}^{n-1}_{R/K}(R,R/P))$.

Conversely, if $r \in (0 :_R \mathrm{D}^{n-1}_{R/K}(R,R/P))$, and $\delta \in \mathrm{D}^{n-1}_{R/K}$, then $(\pi\circ\delta) \in \mathrm{D}^{n-1}_{R/K}(R,R/P)$, so $(\pi \circ \delta)(r) = 0$, which means $\delta(r) \in P$. Thus, $(0 :_R \mathrm{D}^{n-1}_{R/K}(R,R/P)) \subseteq (P :_R \mathrm{D}^{n-1}_{R/K})$. □

Chapter 4

Examples of rings of differential operators

We now want to compute to actual rings of differential operators on rings other than polynomial rings.

Exercise 4.0.1. *Let K be a field, and R be module finite K-algebra. Then,* $\mathrm{D}_{R/K} = \mathrm{Hom}_K(R, R)$.

We will develop a few methods/ tricks to deal with these computations. First, we note:

Proposition 4.0.2. *Let R be a graded A-algebra, with $A \subseteq [R]_0$, and grading group G. Then, $\mathrm{D}_{R/A}$ is a G-graded A-algebra.*

Proof. We observe that $R \otimes_A R$ admits a G-grading by setting $[R \otimes_A R]_c = \bigoplus_{a+b=c} [R]_a \otimes_A [R]_b$. The ideal $\Delta_{R/A}$ is homogeneous, as it is generated by homogeneous elements $\{r \otimes 1 - 1 \otimes r \mid r \text{ homogeneous}\}$. Thus, $\mathrm{P}^i_{R/A}$ is graded, and this is comparable with the grading on R, so it is a graded R-module. Then, $\mathrm{D}^i_{R/A} \simeq \mathrm{Hom}_R(\mathrm{P}^i_{R/A}, R)$ is a graded R-module for each i. We claim that $\delta \in \mathrm{D}^i_{R/A}$ is homogeneous of degree t iff $\delta([R]_a) \subseteq [R]_{a+t}$ for each a. Indeed, write $\delta = \varphi \circ d_R$, $\varphi : \mathrm{P}^i_{R/A} \longrightarrow R$ of degree t. Then if $r \in [R]_a$, $\delta(r) = \varphi(1 \otimes r)$ has degree $a + t$, since $\deg(1 \otimes r) = a$. The converse is the same. It is clear that this grading is compatible with multiplication. □

Now, we recall that any differential operator extends uniquely to a localization. Suppose R is a ring (A-algebra), $W \subseteq R$ multiplicative closed set with no zerodivisor, so $R \subseteq W^{-1}R$. Then a map δ in $\mathrm{D}^i_{W^{-1}R/A}$ is an estension of an operator in $\mathrm{D}^i_{R/A}$ iff $\delta(R) \subseteq R$. Put together,

$$\{\delta \in \mathrm{D}^i_{W^{-1}R/A} \mid \delta(R) \subseteq R\} \xrightarrow{\sim} \mathrm{D}^i_{R/A}$$

the restriction map.

4.1 The cuspidal plane cubic

Let K be a field of characteristic 0, and $R = K[x^2, x^3] \subseteq S = K[x]$. Let $T = K[x, x^{-1}]$. Observe that $Rx^2 = S_x = T$. Thus,

$$\mathrm{D}^i_{T/K} = \{1, \overline{x}^{-1}, \overline{x}^{-2}, \dots\} \cdot \mathrm{D}^i_{S/K} = \{\overline{x}^n \partial^m \mid n \in \mathbb{Z},\ m \leq i\}.$$

Moreover, $\mathrm{D}_{R/K}$, $\mathrm{D}_{T/K}$ are graded, and

$$[\mathrm{D}_{T/K}]_t = K \cdot \{\overline{x}^{m+t}\partial^m \mid m \geq 0\}.$$

By previous discussion,

$$\mathrm{D}_{R/K} = \{\delta \in \mathrm{D}^i_{T/K} \mid \delta(R) \subseteq R\},$$

and this preserves order and grading.

Since $x^2 S \subseteq R$, we have $\overline{x}^2\,\mathrm{D}_{S/K} \subseteq \mathrm{D}_{R/K}$ i.e., $K \cdot \{\overline{x}^n\partial^m \mid n \geq 2,\ m \geq 0\} \subseteq \mathrm{D}_{R/K}$. $[\mathrm{D}_{R/K}]_t \supseteq [\mathrm{D}_{S/K}]_{t-2}$. $\overline{x}^2[\mathrm{D}_{S/K}]_{t-2} = K \cdot \{\overline{x}^{m+t}\partial^m \mid m \geq 0,\ m+t \geq 2\}$.

We will compute $[\mathrm{D}_{R/K}]_t$ for various degrees t.

$t \geq 2$) In this case, $[\mathrm{D}_{T/K}]_t = \overline{x}^2[\mathrm{D}_{S/K}]_{t-2}$, so $[\mathrm{D}_{R/K}]_t = [\mathrm{D}_{T/K}]_t$.

$t = 1$) $[\mathrm{D}_{T/K}]_1 = K \cdot \{\overline{x}\} + \overline{x}^2[\mathrm{D}_{S/K}]_{-1}$. No nonzero element of $K\{\overline{x}\}$ stabilizes R: $(\lambda\overline{x})(1) = \lambda x \notin R$.

$t = 0$) $[\mathrm{D}_{T/K}]_0 = K \cdot \{\overline{1}, \overline{x}\partial\}\overline{x}^2 + [\mathrm{D}_{S/K}]_{-2}$. Both $\overline{1}$ and $\overline{x}\partial$ stabilize R.

$t = -1$) $[\mathrm{D}_{T/K}]_{-1} = K \cdot \{\overline{x}^{-1}, \partial, \overline{x}\partial^2\} + \overline{x}^2[\mathrm{D}_{S/K}]_{-3}$. Note that these operators send $[R]_{\geq 3}$ into R. So, we check $1, x^2 \dots$ find that it is spanned by $\partial - \overline{x}\partial^2$.

$t = -2$) Similarly, $[\mathrm{D}_{R/K}]_{-2} = K \cdot \{2\overline{x}^{-1}\partial - \partial^2\} + \overline{x}^2[\mathrm{D}_{S/K}]_{-4}$.

$t < -2$) $[\mathrm{D}_{T/K}]_t = K \cdot \{\overline{x}^t, \overline{x}^{t+1}\partial, \dots, \overline{x}\partial^{-t+1}\} + \overline{x}^2[\mathrm{D}_{S/K}]_{t-2}$. It suffices to check theses operators sent $1, x^2.x^3, \dots, x^{-t+1}$ into R. One gets a system of linear equations, and one can check that for each such t, there is a 2-dim K-vector space V_t such that $[\mathrm{D}_{R/K}]_t = V_t + \overline{x}^2[\mathrm{D}_{S/K}]_{t-2}$. In particular, one finds $3\overline{x}^{-2}\partial - 3\overline{x}^{-1}\partial^2 + \partial^3 \in [\mathrm{D}_{R/K}]_{-3}$.

One can verify that

$$\mathrm{D}_{R/K} = \overline{R}\left\langle \overline{x}\partial, \overline{x}^2\partial, \overline{x}\partial^2 - \partial, \partial^2 - 2\overline{x}^{-1}\partial, \partial^3 - 3\overline{x}^{-1}\partial^2 + 3\overline{x}^{-2}\partial \right\rangle \subseteq \mathrm{D}_{T/K},$$

by checking this generates $\mathrm{D}_{R/K}$ in positive degrees, and that this subring quotiented by $\overline{x}^2[\mathrm{D}_{S/K}]$ is 2-dim in each degree.

4.2 Invariants of finite groups

Our next family of examples is invariants of finite groups. Throughout the setting is as follows:

- K a field,
- $R = K[x_1, \ldots, x_n]$ polynomial ring,
- G finite group, with order $|G|$ nonzero in K, acting linearly on R.

Note that G acts on $\mathrm{Spec}(R)$: $\sigma(\mathfrak{p})$ is a prime (maximal) for $\mathfrak{p}$ prime (maximal) $\sigma \in G$. Then,

$$\mathrm{Max}_K(R) \subseteq \mathrm{Max}(R) \subseteq \mathrm{Spec}(R),$$

where

$$\begin{aligned}\mathrm{Max}_K(R) &:= \{\mathfrak{m} \text{ maximal with } R/\mathfrak{m} \cong K\}\\ &= \{(x_1 - a_1, \ldots, x_n - a_n) = \mathfrak{m}_{\underline{a}} \mid \text{for } \underline{a} \in K^n\}\\ &\simeq K^n.\end{aligned}$$

If $K = \overline{K}$ then $\mathrm{Max}(R) = \mathrm{Max}_K(R)$. G acts on $\mathrm{Spec}(R)$, on $\mathrm{Max}(R)$, $\mathrm{Max}_K(R) \simeq K^n$

$$g \cdot \mathfrak{m}_{\underline{a}} = (Ax_1 - a_1, \ldots, Ax_n - a_n)$$

so

$$A \cdot \underline{x} = \underline{a} \leftrightsquigarrow \underline{x} = A^{-1}\underline{a}$$

then,

$$g \cdot \mathfrak{m}_{\underline{a}} = \mathfrak{m}_{A^{-1}\underline{a}}$$

(right action of G).

For $\sigma \in G$, we have $\mathrm{Fix}(\sigma) = \{\mathfrak{n} \in \mathrm{Max}(R) \mid \sigma(\mathfrak{n}) \subseteq \mathfrak{n}\}$, and $\mathrm{Fix}_K(\sigma) = \mathrm{Fix}(\sigma) \cap \mathrm{Max}_K(R)$. In addition, for $\mathfrak{n} \in \mathrm{Max}(R)$, we have $\mathrm{Stab}(\mathfrak{n}) = \{\sigma \in G \mid \sigma(\mathfrak{n}) = \mathfrak{n}\}$.

Proposition 4.2.1. *Let $K = \overline{K}$. Let $\mathfrak{n} \in \mathrm{Max}(R)$ be such that $\mathrm{Stab}(\mathfrak{n}) = \{e\}$, and let $\mathfrak{m} = \mathfrak{n}^G = \mathfrak{n} \cap R^G$. Then, $R^G/\mathfrak{m}^n \simeq R/\mathfrak{n}^n$ for all $n \in \mathbb{N}$.*

Proof. Since R^G and R are finitely generated algebras over an algebra closed field, we have $\mathfrak{m} \in \mathrm{Max}(R^G)$, and $R^G/\mathfrak{m} \simeq K \simeq R/\mathfrak{n}$.

We claim that $\mathfrak{m}R_{\mathfrak{m}} = \mathfrak{n}R_{\mathfrak{n}}$. To see this, let $J = (\{f \in \mathfrak{n} \mid f \notin \sigma(\mathfrak{n})\ \forall \sigma \neq e\})$. Then,

$$\mathfrak{n} = J \cup \left(\bigcup_{\sigma \neq e} \sigma(\mathfrak{n}) \right).$$

Each $\sigma(\mathfrak{n})$ is a maximal ideal distinct from $\mathfrak{n}$ (by $\mathrm{Stab}(\mathfrak{n}) = \{e\}$), so $\mathfrak{n} \not\subseteq \sigma(\mathfrak{n})$, and by prime avoidance, we must have $\mathfrak{n} \subseteq J$, so $\mathfrak{n} = J$. Thus, we can write $\mathfrak{n} = (f_1, \dots, f_t)$ with $\sigma(f_i) \notin \mathfrak{n}$ for each $\sigma \neq e$, each i. Then, $g_i = \prod_{\sigma \neq \mathfrak{n}} \sigma(f_i) \notin \mathfrak{n}$, so g_i are units in $R_{\mathfrak{n}}$. Thus, $\mathfrak{n} = (g_1 f_1, \dots, g_t f_t)$, and each $g_i f_i \in R^G \cap \mathfrak{n} = \mathfrak{m}$, justifying the claim. (Exercise: The equality $\mathfrak{m}R = \bigcap_{\sigma \in G} \sigma(\mathfrak{n})$ holds in this setting.)

Now, the map $R^G/\mathfrak{m}^n \longrightarrow R/m^n$ is module-finite (the target is even finite length) so the cokernel $M = R_{\mathfrak{n}}/(\mathrm{Im}(R^G) + \mathfrak{n}^n)$ is as well. But $M/\mathfrak{m}M \simeq R_{\mathfrak{n}}/(\mathfrak{m}R_{\mathfrak{n}} \mathrm{Im}(R^G) + \mathfrak{n}^n) \simeq 0$, so $M = 0$ by NAK. Thus, the map is surjective.

For injectivity, we need to show that $\mathfrak{n}^n \cap R^G \subseteq \mathfrak{m}^n$. Note that if $I = (a_1, \dots, a_s) \subseteq R$ and $f \in I \cap R^G$, we have $f = \sum a_i r_i \rightsquigarrow f = \sigma(f) = \sum \sigma(a_i)\sigma(r_i) \in \sigma(I)$ for $\sigma \in G$. Thus, by Chinese Remainder Theorem, we have

$$\begin{aligned}
\mathfrak{n}^n \cap R^G \subseteq \bigcap_{\sigma \in G} \sigma(\mathfrak{n}^n) &= \prod_{\sigma \in G} \sigma(\mathfrak{n}^n) \\
&= \prod_{\sigma \in G} \sigma(\mathfrak{n})^n = \left(\prod_{\sigma \in G} \sigma(\mathfrak{n}) \right)^n \\
&= \left(\bigcap_{\sigma \in G} \sigma(\mathfrak{n}) \right)^n = (\mathfrak{m}R)^n \\
&= \mathfrak{m}^n R,
\end{aligned}$$

so $\mathfrak{n}^n \cap R^G \subseteq \mathfrak{m}^n R \cap R^G = \mathfrak{m}^n$, since R^G is a direct summand of R. □

There is a natural map

$$\frac{R \otimes_K R^G}{\Delta^n_{R^G/K}(R \otimes_K R^G)} \simeq R \otimes_{R^G} \mathrm{P}^n_{R^G/K} \xrightarrow{\alpha} \mathrm{P}^n_{R/K} \simeq \frac{R \otimes_K R}{\Delta^n_{R/K}},$$

since $\Delta_{R^G/K}(R \otimes_K R^G) \subseteq \Delta_{R/K}$.

Proposition 4.2.2. *Let $K = \overline{K}$. Let $\mathfrak{n} \in \mathrm{Max}(R)$ be such that $\mathrm{Stab}(\mathfrak{n}) = \{e\}$, and let $\mathfrak{m} = \mathfrak{n} \cap R^G$. Then, the localization $\alpha_{\mathfrak{n}} : R_{\mathfrak{n}} \otimes_{R^G_{\mathfrak{m}}} \mathrm{P}^n_{R^G_{\mathfrak{m}}/K} \longrightarrow \mathrm{P}^n_{R_{\mathfrak{n}}/K}$ is an isomorphism.*

Proof. First, we check surjectivity. Note that $\mathrm{P}^n_{R_{\mathfrak{n}}/K}$ is finitely generated, so by NAK, it suffices to see that $\mathrm{P}^n_{R_{\mathfrak{n}}/K} = \mathrm{Im}(\alpha_{\mathfrak{n}}) + \mathfrak{n}\, \mathrm{P}^n_{R_{\mathfrak{n}}/K}$. But,

$$\frac{\mathrm{P}^n_{R_{\mathfrak{n}}/K}}{\mathfrak{n}\, \mathrm{P}^n_{R_{\mathfrak{n}}/K}} \simeq K \otimes_{R_{\mathfrak{n}}} \mathrm{P}^n_{R_{\mathfrak{n}}/K} \simeq \frac{R_{\mathfrak{n}}}{\mathfrak{n}^{n+1}} \simeq \frac{R^G_{\mathfrak{m}}}{\mathfrak{m}^{n+1}} \simeq \frac{R_{\mathfrak{n}} \otimes_{R^G_{\mathfrak{m}}} \mathrm{P}^n_{R^G_{\mathfrak{m}}/K}}{\mathfrak{n}(R_{\mathfrak{n}} \otimes_{R^G_{\mathfrak{m}}} \mathrm{P}^n_{R^G_{\mathfrak{m}}/K})},$$

verifying surjectivity.

To see injectivity, note that $\mathrm{P}^n_{R_\mathfrak{n}/K}$ is local with maximal ideal $(\mathfrak{n}\otimes 1+1\otimes\mathfrak{n})$:

$$\operatorname{Spec}\left(\frac{R_\mathfrak{n}\otimes_K R_\mathfrak{n}}{\Delta^{n+1}_{R_\mathfrak{n}/K}}\right)\simeq \operatorname{Spec}\left(\frac{R_\mathfrak{n}\otimes_K R_\mathfrak{n}}{\Delta_{R_\mathfrak{n}/K}}\right)\simeq \operatorname{Spec}(R_\mathfrak{n}),$$

which is local; similarly, $R_\mathfrak{n}\otimes_{R^G_\mathfrak{m}}\mathrm{P}^n_{R^G_\mathfrak{m}/K}$ is local with maximal ideal $(\mathfrak{n}\otimes 1+1\otimes\mathfrak{m})$.

Let α_t be the map $\frac{R_\mathfrak{n}\otimes_{R^G_\mathfrak{m}}\mathrm{P}^n_{R^G_\mathfrak{m}/K}}{(\mathfrak{n}^t\otimes 1+1\otimes\mathfrak{m}^t)}\longrightarrow\frac{\mathrm{P}^n_{R_\mathfrak{n}/K}}{(\mathfrak{n}^t\otimes 1+1\otimes\mathfrak{n}^t)}$. An element in $R_\mathfrak{n}\otimes_{R^G_\mathfrak{m}}\mathrm{P}^n_{R^G_\mathfrak{m}/K}$ is in $\operatorname{Ker}(\alpha_\mathfrak{n})$ iff its image is in $\operatorname{Ker}(\alpha_t)$ for each t, since $\bigcap_t(\mathfrak{n}^t\otimes 1+1\otimes\mathfrak{n}^t)\subseteq\bigcap_t(\mathfrak{n}\otimes 1+1\otimes\mathfrak{n})^t=0$, by Krull intersection. But the isomorphism $R^G_\mathfrak{m}/\mathfrak{m}^t\simeq R_\mathfrak{n}/n^t$ induces isomorphisms α_t fo every t. Thus, $\operatorname{Ker}(\alpha_\mathfrak{n})\subseteq\bigcap_t(\mathfrak{n}^t\otimes 1+1\otimes\mathfrak{m}^t)\subseteq\bigcap_t(\mathfrak{n}\otimes 1+1\otimes\mathfrak{m})^t=0$, so $\alpha_\mathfrak{n}$ is also injective. □

Applying $\operatorname{Hom}_R(-,R)$, α induces a map

$$\begin{array}{ccc}\operatorname{Hom}_R(\mathrm{P}^n_{R/K},R) & \longrightarrow & \operatorname{Hom}_R(R\otimes_{R^G}\mathrm{P}^n_{R^G/K},R)\simeq\operatorname{Hom}_{R^G}(\mathrm{P}^n_{R^G/K},R)\\ \sim\downarrow d^* & & \sim\downarrow d^*\\ \mathrm{D}^n_{R/K} & \xrightarrow{\ \beta\ } & \mathrm{D}^n_{R^G/K}(R^G,K)\end{array}$$

Exercise 4.2.3. *β is just the restriction map.*

It follows that $\beta_\mathfrak{n}$ is an isomorphism for all $\mathfrak{n}\in\operatorname{Max}(R)$ with trivial stabilizer.

Let $\sigma\in G$, and $\underline{a}\in K^n$. Then,

$$\begin{aligned}\mathfrak{m}_{\underline{a}}\in\operatorname{Fix}_K(\sigma) &\Leftrightarrow(\underline{x}-\underline{a})=(\underline{\sigma(x)}-\underline{a}\\ &\Leftrightarrow(\underline{x}-\underline{\sigma(x)})\in\mathfrak{m}_{\underline{a}},\end{aligned}$$

so $\operatorname{Fix}_K(\sigma)=\mathcal{V}(x_1-\sigma(x_1),\dots,x_n-\sigma(x_n))\cap\operatorname{Max}_K(R)$, which is a linear subspace of $\operatorname{Max}_K(R)\simeq K^n$ of codimension equal to the rank of $(id-\sigma)$ as a linear transformation on $[R]_1$.

Definition 4.2.4. We say that $\sigma\in G\backslash\{e\}$ is a **pseudoreflection** if $\operatorname{Fix}_K(\sigma)\subseteq\operatorname{Max}_K(R)\simeq K^n$ has codimension one.

Proposition 4.2.5. *Let $K=\overline{K}$. If G contain no pseudoreflections, then the restriction map $\mathrm{D}^n_{R/K}\xrightarrow{\beta}\mathrm{D}^n_{R^G/K}(R^G,R)$ is an isomorphism.*

Proof. If $\dim(R)=1$, then the condition implies $G=\{e\}$, so $R^G=R$ and this is trivial. WLOG $\dim(R)=2$.

Note first that $\mathrm{D}^n_{R/K}$ is a free R-module, and that $\mathrm{D}^n_{R^G/K}(R^G,R)\simeq\operatorname{Hom}_{R^G}(\mathrm{P}^n_{R^G/K},R)$ is a torsion free R-module, since postmultiplying by an element in R, which is torsion free, cannot kill nonzero map.

We claim now that if $\mathrm{ht}(\mathfrak{p}) \leq 1$ in R, then $\beta_{\mathfrak{p}}$ is an isomorphism. To see this, note that the hypothesis means that $\mathcal{V}(\mathfrak{p}) \not\subseteq \bigcup_{\sigma\neq e} \mathrm{Fix}(\sigma)$. As $\mathfrak{p}$ is an intersection of maximal ideals, there exists $\mathfrak{n} \in \mathcal{V}_{\mathrm{Max}}(\mathfrak{p}) \setminus \bigcup_{\sigma\neq e} \mathrm{Fix}(\sigma)$. That is, there is a maximal ideal $\mathfrak{n}$ with $\mathfrak{p} \subseteq \mathfrak{n}$ and $\mathrm{Stab}(\mathfrak{n}) = \{e\}$. Thus, $\beta_{\mathfrak{n}}$ is an isomorphism, and localizing further, $\beta_{\mathfrak{p}}$ is an isomorphism as well.

To see that β is injective, we have $\mathrm{Ass}_R(\mathrm{Ker}(\beta)) \subseteq \mathrm{Ass}_R(\mathrm{D}^n_{R/K}) = (0)$, but $\beta_{(0)}$ is injective, so $\mathrm{Ass}_R(\mathrm{Ker}(\beta)) = \emptyset$, and β is injective. Since $\mathrm{Hom}_{R^G}(\mathrm{P}^n_{R^G/K}, R)$ is torsion free, $\mathrm{depth}_{R_{\mathfrak{p}}}(\mathrm{Hom}_{R^G}(\mathrm{P}^n_{R^G/K}, R)_{\mathfrak{p}}) \geq 1$ for all primes $\mathfrak{p} \neq 0$. Then if $\mathrm{ht}(\mathfrak{p}) \leq 1$, $\mathrm{Coker}(\beta)_{\mathfrak{p}} = 0$, and if $\mathrm{ht}(\mathfrak{p}) \geq 2$, $\mathrm{depth}(\mathrm{D}^n_{R/K}) = \mathrm{ht}(\mathfrak{p}) \geq 2$, since $(\mathrm{D}^n_{R/K})_{\mathfrak{p}}$ is free over R_{fp} (which is Cohen-Macaulay). Thus, if $\mathrm{Coker}(\beta)_{\mathfrak{p}} \neq 0$, by the behavior of depth on SES's, we have $\mathrm{depth}(\mathrm{Coker}(\beta)_{\mathfrak{p}}) \geq 1$. But, if $\mathrm{Coker}(\beta) \neq 0$, there exist $\mathfrak{q} \in \mathrm{Ass}_R(\mathrm{Coker}(\beta))$, and $\mathrm{depth}(\mathrm{Coker}(\beta)_{\mathfrak{q}}) = 0$. Thus, $\mathrm{Coker}(\beta) = 0$, so β is an isomorphism. □

Proposition 4.2.6. *Without assuming $K = \overline{K}$, if G has no pseudoreflections, then the restriction map $\mathrm{D}^n_{R/K} \xrightarrow{\beta} \mathrm{D}^n_{R^G/K}(R^G, R)$ is an isomorphism.*

Proof. It suffices to show that $\mathrm{D}^n_{R/K} \otimes_K \overline{K} \xrightarrow{\beta\otimes_K\overline{K}} \mathrm{D}^n_{R^G/K}(R^G, R) \otimes_K \overline{K}$ is an isomorphism, since $K \longrightarrow \overline{K}$ is faithfully flat.

Let $R_{\overline{K}} = R \otimes_K \overline{K}$, polynomial ring over $\overline{K}$. Then G acts on $R_{\overline{K}}$ by $\sigma(r \otimes \lambda) := \sigma(r) \otimes \lambda$. Observe that the action of G on R has no pseudorefletions if and only if the action of G on $R \otimes_K \overline{K}$ has none, since the rank of $(id - \sigma)$ on 1-forms is the same.

We will show that $\beta \otimes_K \overline{K}$ identifies with $\mathrm{D}^n_{R_{\overline{K}}/\overline{K}} \xrightarrow{\beta_{\overline{K}}} \mathrm{D}^n_{R^G_{\overline{K}}/\overline{K}}(R^G_{\overline{K}}, R_{\overline{K}})$, and the we will be done by Proposition 4.2.5.

First, note that we have a left-exact sequence

$$0 \longrightarrow R^G \longrightarrow R \xrightarrow{\begin{bmatrix} id - \sigma_1 \\ \vdots \\ id - \sigma_{|G|} \end{bmatrix}} R,$$

$G = \{\sigma_1, \ldots, \sigma_{|G|}\}$, so by flatness we have

$$0 \longrightarrow R^G \otimes_K \overline{K} \longrightarrow R_{\overline{K}} \xrightarrow{\begin{bmatrix} id - \sigma_1 \\ \vdots \\ id - \sigma_{|G|} \end{bmatrix}} R_{\overline{K}},$$

so $R^G \otimes_K \overline{K} \simeq (R_{\overline{K}})^G$ canonically. Then, is S is any finitely generated K-algebra, we have $\mathrm{P}^n_{S/K} \otimes_K \overline{K} \simeq \mathrm{P}^n_{S\otimes\overline{K}/\overline{K}}$, and by the behavior of Hom and flat base change, and finite presentation of $\mathrm{P}^n_{S/K}$, we obtain $\mathrm{D}^n_{S/K} \otimes_K \overline{K} \simeq \mathrm{D}^n_{S\otimes\overline{K}/\overline{K}}$. Applying these two observations, we obtain the desired identification. □

Theorem 4.2.7 (Kantor)**.** *Let G be a finite group acting linearly on a polynomial ring R over a field K, with $|G| \neq 0$ in K. Assume G contains no pseudoreflections. Then, the restriction map*

$$\left\{\delta \in \mathrm{D}^n_{R/K} \mid \delta(R^G) \subseteq R^G\right\} \longrightarrow \mathrm{D}^n_{R^G/K}$$

is an isomorphism. Moreover, we have the equality

$$\left\{\delta \in \mathrm{D}^n_{R/K} \mid \delta(R^G) \subseteq R^G\right\} = (\mathrm{D}^n_{R/K})^G,$$

where G acts on $\mathrm{D}^n_{R/K}$ via $g \cdot \delta = g \circ \delta \circ g^{-1}$.

Proof. There are maps

$$\mathrm{D}^n_{R^G/K} \xrightarrow{i} \mathrm{D}^n_{R^G/K}(R^G, R) \xrightarrow{\sim} \mathrm{D}^n_{R/K}.$$

The second map has inverse $\longleftarrow$ given by restriction. The first is injective, since there is an inverse $\longleftarrow$ $\delta \mapsto \frac{1}{|G|}\sum_{g\in G} g \circ \delta$. We see that the composition $\to\to$ has image equal to the maps sending R^G into R^G, and any such map goes via $\leftarrow\leftarrow$ to this restriction.

For the second claim, if $\delta \in (\mathrm{D}^n_{R/K})^G$, and $r \in R^G$, then $g(\delta(r)) = (g\cdot\delta)(g(r)) = \delta(r)$, so $\delta(r) \in R^G$, thus $(\mathrm{D}^n_{R/K})^G \subseteq \left\{\delta \in \mathrm{D}^n_{R/K} \mid \delta(R^G) \subseteq R^G\right\}$. For other contention, let $\delta(R^G) \subseteq R^G$, and take $g \in G$. We need to show that $g \cdot \delta - \delta$ is zero in $\mathrm{D}^n_{R/K}$. By the canonical isomorphism above, it suffices to show it is zero on R^G, so let $r \in R^G$. Then $(g \cdot \delta - \delta)(r) = g\delta(g^{-1}(r)) - \delta(r) = g\delta(r) - \delta(r) = 0$, since $\delta(r) \in R^G$. □

Example 4.2.8. Let K be an algebraically closed field, $R = K[x_1, \dots, x_n]$, $n \geq 2$ and d be an integer that is invertible in K (not a multiple of $\operatorname{char} K$, if $\operatorname{char} K = p > 0$). $G \simeq \mathbb{Z}/d$ acts on R by $g \cdot x_i = \mathfrak{g} x_i$ for g a generator of G, and $\mathfrak{g}$ a primitive d^{th} root of unity.

The fixed space of g, or any nonidentity element, is just the origin, so the theorem applies. We have $R^G = R^{(d)} := \bigoplus_{n\in\mathbb{N}} [R]_{nd}$, the d^{th} Veronese subring, which consists of elements whose homogeneous pieces having degrees a multiple of d.

We compute

$$\mathrm{D}_{R^G} = \left\{\delta \in \mathrm{D}_R \mid \delta(R^G) \subseteq R^G\right\}$$

(restriction of) write any $\delta \in \mathrm{D}_R$ as a sum of homogeneous pieces $\delta = \sum_j \delta_j$. Then δ_j of degree j satisfies $\delta_j(R^G) \subseteq R^G$ iff $d|j$. We conclude that

$$\mathrm{D}_{R^G} = \{\delta \in \mathrm{D}_R \mid \text{every homogeneous piece of } \delta \text{ has degree a multiple of } d\}$$

For example, $\mathrm{D}_{R^{(2)}} = R\left\langle \left\{ x_i \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \right\} \right\rangle$ if $\operatorname{char} K = 0$.

Exercise 4.2.9. *Suppose that $g \in G$ acts on R via*

$$g \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Then, the action of g on D_R is given by $(\operatorname{char} K = 0)$

$$g \cdot \begin{pmatrix} \overline{x_1} \\ \vdots \\ \overline{x_n} \\ \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix} = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & (A^T)^{-1} \end{array}\right) \begin{pmatrix} \overline{x_1} \\ \vdots \\ \overline{x_n} \\ \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix}$$

4.3 Differential operators on Stanley-Reisner rings

Our next class of examples is Stanley-Reisner rings. Let $S = K[x_1, \dots, x_n]$ be a polynomial rings over a field K, $I = P_1 \cap \cdots \cap P_n$ be a squarfree monomial ideal, where $P_i = (\{x_j \mid x_j \in S_j\})$ are monomial prime ideals. $R = S/I$ is a Stanley-Reisner ring.

First, we observe a fact about operators preserving ideals.

Proposition 4.3.1. *Let $A \longrightarrow R$ be commutative rings, with R Noetherian. Let $I \subseteq R$ be an ideal, and Q a minimal primary component of I. Then,*

(1) $\delta \in (I :_{\mathrm{D}_{R/A}} I) \Rightarrow [\delta, \overline{r}] \in (I :_{\mathrm{D}_{R/A}} I)$ *for any* $r \in R$.

(2) $(I :_{\mathrm{D}_{R/A}} I) \subseteq (Q :_{\mathrm{D}_{R/A}} Q)$.

Proof. (1) If $a \in I$, then $[\delta, \overline{r}](a) = \delta(ra) - r\delta(a) \in I$.

(2) Observe that the intersection of the other primary components of I (say $\mathfrak{a}_1, \dots, \mathfrak{a}_s$) is not contained in $\sqrt{Q}$: otherwise some $\mathfrak{a}_i \subseteq \sqrt{Q}$, but $\operatorname{Min}(\mathfrak{a}_i) = \{\sqrt{\mathfrak{a}_i}\}$, contradicting minimality. So, we take $f \in (\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_s) \backslash \sqrt{Q}$. Note that $fQ \subseteq I$.

We will shoe that $(I :_{\mathrm{D}^i_{R/A}} I) \subseteq (Q :_{\mathrm{D}^i_{R/A}} Q)$ by induction on i, with base case $i = 0$ trivial.

$\delta \in (Q :_{\mathrm{D}^{i-1}_{R/A}} Q)$ by Part (1) and IH. For $q \in Q$, then $\delta(fq) - f\delta(q) \in Q$. $fq \in I \Rightarrow \delta(fq) \in I \subseteq Q$, so $f\delta(q) \in Q$, hence $\delta(q) \in Q$.

□

Proposition 4.3.2. *Let Q be a monomial ideal prime. Then,*

$$(Q :_{\mathrm{D}_{R/K}} Q) = K \cdot \{x^{\underline{\alpha}} \partial^{(\underline{\beta})} \mid x^{\underline{\alpha}} \in Q \text{ or } x^{\underline{\beta}} \notin Q\}.$$

Proof. For the containment ($\supseteq$), it suffices to check for the basis elements.

If $x^{\underline{\alpha}} \in Q$, $x^{\underline{\alpha}}\partial^{(\underline{\beta})} \in (Q :_{\mathrm{D}_{R/K}} Q)$ is clear.

If $x^{\underline{\beta}} \notin Q$, then $x^{\underline{\alpha}}\partial^{(\underline{\beta})}$ can only decrease exponents of the variables not in Q, so must stabilize Q.

For the other containment, suppose $\delta \in (Q :_{\mathrm{D}_{R/K}} Q)$ is not in the RHS. Subtracting off an element of the RHS, we can assume that all of the terms $\lambda_{\alpha,\beta} x^{\underline{\alpha}}\partial^{(\underline{\beta})}$, $\lambda_{\alpha,\beta} \neq 0$ have $x^{\underline{\alpha}} \notin Q$ and $x^{\underline{\beta}} \in Q$.

Let $(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)$ be the nonzero pairs (α, β) as above in δ. Suppose that β_1 is such that $|\beta_1| \leq |\beta_i|$ all i. Then,

$$\begin{aligned}
\delta(x^{\underline{\beta}_1}) &= \sum_{(\alpha,\beta)} x^{\underline{\alpha}}\partial^{(\underline{\beta})}(x^{\underline{\beta}_1}) \\
&= x^{\underline{\alpha_1}}\partial^{(\underline{\beta_1})}(x^{\underline{\beta}_1}) \\
&= x^{\underline{\alpha_1}},
\end{aligned}$$

but $x^{\underline{\beta}_1} \in Q$, $x^{\underline{\alpha}_1} \notin Q$ contradicts that $\delta \in (Q :_{\mathrm{D}_{R/K}} Q)$. □

Theorem 4.3.3 (Tripp, Traves)**.** *Let R/S be a Stanley-Reisner ring, with $I = P_1 \cap \cdots \cap P_t$ squarfree monomial ideal, P_i primes. Then,*

$$\mathrm{D}_{R/K} = K \cdot \{x^{\underline{\alpha}}\partial^{(\underline{\beta})} \mid x^{\underline{\alpha}} \notin I,\ x^{\underline{\alpha}} \in P_i \text{ or } x^{\underline{\beta}} \notin P_i \text{ for each } i\}$$

as K-vector spaces, with composition induced by the corresponding operators on S.

Proof. We use the description $\mathrm{D}_{R/K} = (I :_{\mathrm{D}_{R/K}} I)/I\,\mathrm{D}_{R/K}$.

To compute $(I :_{\mathrm{D}_{R/K}} I)$, observe first that if $\delta \in \bigcap_i (P_i :_{\mathrm{D}_{R/K}} P_i)$, and $a \in I$, then $a \in P_i$ for each i, so $\delta(a) \in P_i$ for each i, hence $\delta(a) \in I$. Conversely, $(I :_{\mathrm{D}_{R/K}} I) \subseteq \bigcap_i (P_i :_{\mathrm{D}_{R/K}} P_i)$ by Proposition 4.3.1, and the equality holds.

Using the Proposition 4.3.2,

$$(P_i :_{\mathrm{D}_{R/K}} P_i) = K \cdot \{x^{\underline{\alpha}}\partial^{(\underline{\beta})} \mid x^{\underline{\alpha}} \in P_i \text{ or } x^{\underline{\beta}} \notin P_i\}.$$

Then, the given basis above comes from intersecting these and removing those in $I\,\mathrm{D}_{R/K}$ in the monomial basis. □

Example 4.3.4. Let $R = K[x,y]/(xy)$. Writing $(xy) = (x) \cap (y)$, we have

$$\begin{aligned}
\mathrm{D}_{R/K} &= K \cdot \{\overline{x}^a\overline{y}^b\partial^{(c,d)} \mid a \text{ or } b = 0;\ a > 0 \text{ or } c = 0;\ b > 0 \text{ or } d = 0\} \\
&= K \cdot \left(\{\overline{1}\} \cup \{\overline{x}^a\partial^{(c,0)} \mid a \geq 1, c \geq 0\} \cup \{\overline{y}^b\partial^{(0,d)} \mid b \geq 1, d \geq 0\}\right).
\end{aligned}$$

If K has characteristic zero, then we write as

$$K \cdot \left(\{\overline{1}\} \cup \left\{\overline{x}^a \left(\frac{\partial}{\partial x}\right)^c \mid a \geq 1, c \geq 0\right\} \cup \left\{\overline{y}^b \left(\frac{\partial}{\partial y}\right)^d) \mid b \geq 1, d \geq 0\right\}\right).$$

Even in the char 0 case, $\mathrm{D}_{R/K}$ is not a finitely generated K-algebra: in every order i, there is $\bar{x}\left(\frac{\partial}{\partial x}\right)^i \in \mathrm{D}^i_{R/K}$ that is not in the algebra generated by $\mathrm{D}^{i-1}_{R/K}$, thus any algebra generates set must involve arbitrarily high orders, and thus must be infinite.

4.4 Cone over elliptic curve

Let $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$.

Theorem 4.4.1 (Bernstein-Getfand-Gelfand)**.**

(1) $[\mathrm{D}_{R/\mathbb{C}}]_{<0} = 0$*: there are no differential operators of negative degree.*

(2) $[\mathrm{D}_{R/\mathbb{C}}]_0 = \mathbb{C}[E]$ *where* $E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$*. Every operator of degree zero is a polynomial in the Euler operator.*

(3) $\frac{[\mathrm{D}^i_{R/\mathbb{C}}]_1}{[\mathrm{D}^{i-1}_{R/\mathbb{C}}]_1 + E\cdot[\mathrm{D}^{i-1}_{R/\mathbb{C}}]_1} \simeq \mathbb{C}^3$ *for each* i*.*

It follows from these facts that $\mathrm{D}_{R/\mathbb{C}}$ is not a finitely generated $\mathbb{C}$-algebra. Indeed, if we set

$$A_k = [D_{R/\mathbb{C}}]_0 + \sum_{n\geq 0} E^n [D^k_{R/\mathbb{C}}]_1 + [D_{R/\mathbb{C}}]_{\geq 2},$$

then $D^k_{R/\mathbb{C}} \subseteq A_k \subsetneq A_{k+1} \subseteq D_{R/\mathbb{C}}$ for each k, so $D_{R/\mathbb{C}}$ is not generated by $D^k_{R/\mathbb{C}}$ for any k, so is not finitely generated.

We skip the proof of this: one proceeds by analyzing the cohomology of the tangent bundle on $\mathrm{Proj}(R)$ the curve.

4.5 Differential operators in positive characteristic

Theorem 4.5.1. *Let* K *be a perfect field, and* R *be essentially of finite type over* K*. Then,*

$$\mathrm{D}_{R/K} = \bigcup_{e\in\mathbb{N}} \mathrm{Hom}_{R^{p^e}}(R, R),$$

and there is a constant c *such that*

$$\mathrm{D}^{p^e}_{R/K} \subseteq \bigcup_{e\in\mathbb{N}} \mathrm{Hom}_{R^{p^e}}(R, R) \subseteq \mathrm{D}^{cp^e}_{R/K}$$

for all $e \geq 0$*.*

Proof. First we observe that $K = K^{p^e} \subseteq R^{p^e}$ for each e, so $\mathrm{Hom}_{R^{p^e}}(R,R) \subseteq \mathrm{Hom}_K(R,R)$ for each e, and both sides of the equality can be considered as subsets of $\mathrm{Hom}_K(R,R)$.

Now, $R \otimes_K R$ is essentially of finite type over K, hence Noetherian, so $\Delta_{R/K}$ is finitely generated, say with c generators. Then, $\Delta_{R/K}^{cp^e} subseteq \Delta_{R/K}^{[p^e]} \subseteq \Delta_{R/K}^{p^e}$ for each e by the pigeonhole principle. Thus, we have

$$\begin{array}{ccccc} (0 :_{\mathrm{Hom}_K(R,R)} \Delta_{R/K}^{cp^e}) & \supseteq & (0 :_{\mathrm{Hom}_K(R,R)} \Delta_{R/K}^{[p^e]}) & \supseteq & (0 :_{\mathrm{Hom}_K(R,R)} \Delta_{R/K}^{p^e}) \\ \parallel & & \parallel & & \parallel \\ \mathrm{D}_{R/K}^{cp^e} & \supseteq & \mathrm{Hom}_{R^{p^e}}(R,R) & \supseteq & \mathrm{D}_{R/K}^{p^e} \end{array}$$

where the middle equality comes from the observation that

$$\begin{aligned} \Delta_{R/K}^{[p^e]} &= \big(\{1 \otimes r^{p^e} - r^{p^e} \otimes 1 \mid r \in R\}\big) \\ &= \big(\{1 \otimes s - s \otimes 1 \mid s \in R^{p^e}\}\big) \\ &= \Delta_{R^{p^e}/K}(R \otimes_K R), \end{aligned}$$

so this ideal kills a map iff the map is R^{p^e}-linear. □

Each $\mathrm{Hom}_{R^{p^e}}(R,R) \subseteq \mathrm{D}_{R/K}$ is a subring, which we often denote $\mathrm{D}_R^{(e)}$, and call differential operators of **level** e.

IT is often advantageous to consider $\mathrm{D}_{R/K}$ as a union of the rings $\mathrm{D}_R^{(e)}$ in positive characteristic; note in particular that each $\mathrm{D}_R^{(e)}$ is a finitely generated R-module, since each $\mathrm{D}_R^{cp^e}$ is and $\mathrm{D}_R^{(e)} \subseteq \mathrm{D}_R^{cp^e}$.

On the other hand, the fact that $\mathrm{D}_{R/K}$ is a union of subalgebras comparable with the order filtration easily implies that $\mathrm{D}_{R/K}$ is not finitely generated K-algebra. Indeed, if it were, there would exist generating set in $\mathrm{D}_R^{(e)}$ for some e, but this generating set would generate a subting of $\mathrm{D}_R^{(e)}$, which is a contradiction.

In particular, no analogous filtration (by subalgebras) exists for any finitely generated algebra of differential operators, specifically, for the polynomial ring in characteristic zero.

Example 4.5.2. Let $R = K[x_1, \ldots, x_n]$ polynomial ring where K is a perfect field of characteristic $p > 0$. Then, $R^{p^e} = K[x_1^{p^e}, \ldots, x_n^{p^e}]$, and

$$R = \bigoplus_{0 \le \alpha_1, \ldots, \alpha_n < p^e} R^{p^e} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

as R^{p^e}-modules (collect monomials via congruence classes of exponents module p). Then, $\mathrm{D}_R^{(e)}$ is a free R^{p^e}-module with basis $\{\varphi_{\alpha,\beta} \mid 0 \le \alpha_i, \beta_i \le p^e - 1\}$ where

$$\varphi_{\alpha,\beta}(x_1^{\sigma_1} \cdots x_n^{\sigma_n}) = \begin{cases} \underline{x}^{\beta} & \text{if } \underline{\sigma} = \underline{\alpha} \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq \sigma_1, \ldots, \sigma_n < p^e$. As ring $\mathrm{D}_R^{(e)} \simeq \mathrm{Mat}_{p^e n \times p^e n}(R^{p^e})$.

On other hand,

$$\mathrm{D}_{R/K} = \bigoplus_{\alpha} \overline{R} \partial^{(\alpha)}$$

so, can write $\frac{\partial}{\partial x_1}$ as a matrix, realizing it in $\mathrm{Hom}_{R^p}(R, R) \simeq \mathrm{Mat}_{p^n \times p^n}(R^p)$, or a different matrix in $\mathrm{Hom}_{R^{p^2}}(R, R) \simeq \mathrm{Mat}_{p^2 n \times p^2 n}(R^{p^2})$.

Conversely, any R^{p^e}linear map from $R \longrightarrow R$ can be written as an $\overline{R}$-linear combination of $\{\partial^{(\alpha)}\}$.

For example, consider $\Phi := \varphi_{(p^e-1,\ldots,p^e-1),(0,\ldots,0)}$, R^{p^e}- linear map sending

$$\begin{aligned} x^{(p^e-1)\cdot \underline{1}} &\longrightarrow 1 \\ \text{others} &\longrightarrow 0. \end{aligned}$$

We have that $\Phi = \partial^{(p^e-1,\ldots,p^e-1),(0,\ldots,0)}$. To see it, write, $\underline{\alpha} = p^e \beta + \underline{\gamma}$ with $0 \leq \gamma_i < p^e$. Then,

$$\Phi(x^{\underline{\alpha}}) = x^{p^e \underline{\beta}} \Phi((x^{\underline{\sigma}}) = \begin{cases} \underline{x}^{p^e \underline{\beta}} & \text{for each } \sigma_i = p^e - 1 \\ 0 & \text{otherwise} \end{cases}$$

and $\partial^{(p^e-1,\ldots,p^e-1)}(\underline{x}^{\underline{\alpha}}) = \begin{pmatrix} \alpha_1 \\ p^e - 1 \end{pmatrix} \cdots \begin{pmatrix} \alpha_n \\ p^e - 1 \end{pmatrix} x^{\underline{\alpha} - (p^e-1)\cdot \underline{1}}$ and check that

$$\begin{pmatrix} a \\ p^e - 1 \end{pmatrix} \equiv \begin{cases} 1 \bmod p & \text{if } a \equiv p^e - 1 \bmod p \\ 0 \bmod p & \text{if otherwise} \end{cases}$$

(Exercise).

Chapter 5

D-algebra simplicity, D-module simplicity, and singularities

5.1 D-modules and D-simplicity

A D-**module** is just a left $\mathrm{D}_{R/A}$-module (for some $A \longrightarrow R$); we often just say D-module when the context $A \longrightarrow R$ is clear.

We say that an R-module $M$is a D-module if M is a left $\mathrm{D}_{R/A}$-module, and the R-module structure on M obtained by restriction of scalars via $R \longrightarrow \mathrm{D}_{R/A}$ agrees with the given R-module structure. That is, $\overline{r}\cdot m = r\cdot m$ for all $r \in R$ $m \in M$, where "$\cdot$" is the $\mathrm{D}_{R/A}$-action and "$\cdot$" is the R-action. It follows from the definition of $\mathrm{D}_{R/A}$ that the ring R is always a D-module.

Example 5.1.1. Quotient rings? Let $R = S/I$, S polynomial ring.

$$\mathrm{D}_{R/K} \simeq \frac{(I :_{D_{S/K}} I)}{I D_{S/K}}.$$

For $R = S/I$ to be a $\mathrm{D}_{R/A}$-module, we need any $\delta \in \mathrm{D}_{R/A}$ to take I into I; that is $(I :_{\mathrm{D}_{R/A}} I) = \mathrm{D}_{R/A}$.

We saw that if $R = S/I$ is a Stanley-Reisner ring, $I = P_1, \dots, P_t$, P_i monomial ideals, then each $(P_i :_{\mathrm{D}_{R/A}} P_i) = \mathrm{D}_{R/A}$. Thus, in this case each P_i or each R/P_i is a D-module.

Definition 5.1.2. $A \longrightarrow R$ commutative ring. A D-**ideal** is a D-submodule of R. Since we have $R \hookrightarrow \mathrm{D}_{R/A}$, a D-ideal must be an ideal of R.

Lemma 5.1.3. *Let I, J be* D*-ideals. Then, $I + J$, $I \cap J$, and every minimal primary component of I (if R Noetherian) is a* D*-ideal.*

Proof. For "+" and "∩" this is clear. If Q is a minimal primary component of I, then $(Q :_{D_{R/A}} Q) = (I :_{D_{R/A}} I) = \mathrm{D}_{R/A}$. □

Remark 5.1.4. If I is a D-ideal, R/I is a D-module.

A key source of D-modules is by localization.

Proposition 5.1.5. *Let $A \longrightarrow R$ commutative rings, and M be a* D*-module. Let $W \subseteq R$ be multiplicatively closed. Then, $W^{-1}M$ is a $\mathrm{D}_{R/A}$-module by the rule*

$$\alpha \cdot \frac{m}{w} = \sum_{i=0}^{\mathrm{ord}(\alpha)} \frac{\alpha^{(i)} \cdot m}{w^{i+1}},$$

where $\alpha^{(0)} := \alpha$, $\alpha^{(i+1)} := [\alpha^{(i)}, \overline{w}]$.

Proof. We will use ?⋆? to denote the function $\mathrm{D}_{R/A} \times W^{-1}M \xrightarrow{\star} W^{-1}M$ defined inductively on the order of an input in $\mathrm{D}_{R/A}$ by the rule

$$\alpha \star m/w := 1/w(\alpha \cdot m - [\alpha, \overline{w}] \star m/w),$$

where "·" denotes the given action $\mathrm{D}_{R/A} \times M \xrightarrow{\cdot} M$. Then, $\mathrm{D}_{R/A} \times W^{-1}M \xrightarrow{\star} W^{-1}M$ is well-defined, and evidently bilinear. It is a trivial induction to verify that $\star$ then agree with the action function specified in the statement. We need to check that is $\mathrm{D}_{R/A}$-linear though (i.e., respects compositions). Hence for then, when an element $w \in W$ is obvious, we write $\alpha' := [\alpha, \overline{w}]$.

First, we observe that if $\overline{r} \in \mathrm{D}^0_{R/A}$ then $\overline{r}$ acts by multiplication on $W^{-1}M$. Now we check $\alpha \star (\beta \star m/w) = (\alpha \circ \beta) \star m/w$ when $\alpha \in \mathrm{D}^0_{R/A}$, then $\beta \in \mathrm{D}^0_{R/A}$.

If $\alpha = \overline{r}$, then

$$\begin{aligned}
\overline{r} \star (\alpha \star m/w) &= r/w(\alpha \cdot m - \alpha' \star m/w) \\
&= 1/w(r(\alpha \cdot m) - r(\alpha' \star m/w)) \\
&= 1/w(\overline{r}(\alpha \cdot m) - (\overline{r}\alpha)' \star m/w) \\
&= (\overline{r} \circ \alpha) \star m/w.
\end{aligned}$$

If $\beta = \overline{r}$, then

$$\begin{aligned}
\alpha \star (\overline{r} \star m/w) &= \alpha \star (rm/w) \\
&= 1/w(\alpha \cdot rm - \alpha' \star rm/w) \\
&= 1/w(\alpha\overline{r} \cdot m - \alpha'\overline{r} \star m/w) \text{ (inductionon } \mathrm{ord}(\alpha)) \\
&= 1/w((\alpha\overline{r}) \cdot m - (\alpha\overline{r})' \star (m/w)) \\
&= (\alpha \circ \overline{r}) \star (m/w).
\end{aligned}$$

By finish by showing $\alpha\star(\beta\star m/w) = (\alpha\circ\beta)\star m/w$ by induction on $\mathrm{ord}(\alpha)+\mathrm{ord}(\beta)$, with the base case of 0 (or 1) already covered. For simplicity, multiply both sides by w. Then,

$$\begin{aligned}
\mathrm{RHS} &= (\alpha\circ\beta)\cdot r - (\alpha\circ\beta)'\star(r/w)\\
&= \alpha\cdot(\beta\cdot r) - (\alpha\cdot\beta')\star(r/w) - (\alpha'\circ\beta)\star(r/w)\\
&= \alpha\cdot(\beta\cdot r) - \alpha\star(\beta'\star(r/w)) - \alpha'\star(\beta\star(r/w)) \text{ (by IH).}
\end{aligned}$$

$$\begin{aligned}
\mathrm{LHS} &= w(\alpha\star(\beta\star(r/w)))\\
&= (\overline{w}\alpha)\star(\beta\star(r/w))\\
&= (\alpha\overline{w}-\alpha')\star(\beta\star(r/w))\\
&= (\alpha\overline{w})\star(\beta\star(r/w)) - \alpha'\star(\beta\star(r/w))\\
&= \alpha\star(\overline{w}\star(\beta\star(r/w))) - \alpha'\star(\beta\star(r/w))\\
&= \alpha\star(\overline{w}\beta\star r/w)) - \alpha'\star(\beta\star(r/w))\\
&= \alpha\star((\beta\overline{w}-\beta')\star(r/w)) - \alpha'\star(\beta\star(r/w))\\
&= \alpha\star(\beta\star\overline{w})\star(r/w) - \alpha\star(\beta'\star(r/w)) - \alpha'\star(\beta\star(r/w))\\
&= \alpha\star(\beta\star r) - \alpha\star(\beta'\star(r/w)) - \alpha'\star(\beta\star(r/w))\\
&= \alpha\cdot(\beta\cdot r) - \alpha\star(\beta'\star(r/w)) - \alpha'\star(\beta\star(r/w))\\
&= \mathrm{RHS}.
\end{aligned}$$

□

In particular, $W^{-1}R$ is always a D-module.

We now observe that the action on any D-module is by differential operators.

Proposition 5.1.6. *Let M be a* D*-module and $\delta \in \mathrm{D}^i_{R/A}$. Then the map*

$$\begin{aligned}
M &\xrightarrow{\delta\cdot} M\\
m &\longrightarrow \delta\cdot m
\end{aligned}$$

is an element of $\mathrm{D}^i_{R/A}(M,M)$.

Proof. By induction on i. For $i=0$, take $\delta=\overline{r}$; clearly $m\longrightarrow rm$ is R-linear, so an element of $\mathrm{Hom}_R(M,M) = \mathrm{D}^0_{R/A}(M,M)$.

For the inductive step, take $\delta\in\mathrm{D}^i_{R/A}$. We need to see that $[(m\to\delta\cdot m),\overline{r}]\in\mathrm{D}^{i-1}_{R/A}(M,M)$ for each $r\in R$. This sends $m\to\delta\cdot(rm)-r(\delta\cdot m)=[\delta,\overline{r}]\cdot m$, which is in $\mathrm{D}^{i-1}_{R/A}(M,M)$ by the inductive hypothesis. □

It is worth noting that, in the noncommutative case annihilators of (left or right) modules are two-side ideals. Indeed, if $\alpha \cdot M = 0$, then $\alpha\beta \cdot M = \alpha \cdot \beta M \subseteq \alpha \cdot M = 0$, and $\beta\alpha \cdot M = \beta \cdot (\alpha \cdot M) = \beta \cdot = 0$. We record this as

Proposition 5.1.7. *The annihilator of a* D*-module is a two-side ideal of* $\mathrm{D}_{R/A}$.

5.2 Local cohomology

If $I \subseteq R$ is an ideal generated by the elements $f_1, \ldots, f_n \in R$. The Čech complex on R of $\underline{f}$ is denoted by $\check{\mathrm{C}}^{\bullet}(\underline{f}; R)$, and defined as:

$$0 \to R \to \bigoplus_i R_{f_i} \to \bigoplus_{i<j} R_{f_i f_j} \to \ldots \to R_{f_1 \cdots f_n} \to 0,$$

and more generally, if M is an R-module, with localizations of M.

The maps are ± 1, with signs chosen in such a way as to make complex.

If $\sqrt{(f_1, \ldots, f_n)} = \sqrt{(g_1, \ldots, g_m)} = I$, then it is a theorem that the cohomologies of the Čech complex agree.

Denote the $H^i_I(M)$, i-th local cohomology of M with support in I.

If M is a D-module, then each $M_{f_{i_1} \cdots f_{i_t}}$ is a D-module, and the maps of localization (up to sign) are clearty D-linear from the formula, we gave. Thus, the Čech complex is a complex of D-modules, so each $H^i_I(M)$ is a D-module as well.

Note that any left ideal of $\mathrm{D}_{R/A}$ is a D-module, and any cyclic D-module is of the from $\mathrm{D}_{R/A}/J$ for some left ideal J.

The left ideal generated by $\delta_1, \ldots, \delta_n$ is

$$\mathrm{D}_{R/A} \cdot \{\delta_i\} = \sum_i \mathrm{D}_{R/A}\, \delta_i = \{\alpha_1 \delta_1 + \ldots + \alpha_n \delta_n \mid \alpha_i \in \mathrm{D}_{R/A}\}.$$

In general, we have

$$0 \longrightarrow J \longrightarrow \mathrm{D}_{R/A} \xrightarrow{\text{evaluate at 1}} R \longrightarrow 0$$

as D-module, where $J = \{\delta \mid \delta(1) = 0\}$. As R-modules this splits

$$\begin{aligned} \mathrm{D}_{R/A} &\rightleftarrows R \\ \overline{r} &\leftarrow r. \end{aligned}$$

Sometimes this J is called the **higher derivations** or the differential operators.

For each i, have $\mathrm{D}^0_{R/A} \subseteq \mathrm{D}^i_{R/A}$, and this splits as R-modules:

$$0 \to \overline{\mathrm{D}}^i_{R/A} \rightleftarrows \mathrm{D}^i_{R/A} \rightleftarrows \mathrm{D}^0_{R/A} \to 0.$$

For $i = 1$, $\overline{\mathrm{D}}^1_{R/A}$ is the A-linear derivations on R, maps that satisfy Leibniz rule $\partial(xy) = x\partial(y) + y\partial(x)$ (Exercise).

Let R be a polynomial ring over A, we have $\mathrm{D}_{R/A} = \bigoplus_\alpha \partial^{(\alpha)}\overline{R}$. Then,

$$\mathrm{Ker}(\mathrm{D}_{R/A} \overset{\text{evaluate at 1}}{\longrightarrow} R)$$

is $J_1 = \mathrm{D}_{R/A} \cdots \{\partial^{(\alpha)} \mid \alpha \neq \underline{0}\}$; so $R \simeq \mathrm{D}_{R/A} / J_1$. In particular, if K is a field of characteristic zero, then $J_1 = \mathrm{D}_{R/K} \cdot \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$, and $R \simeq \mathrm{D}_{R/K} / \mathrm{D}_{R/K} \cdot \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$.

Example 5.2.1. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. Then, $H^n_{(\underline{x})}(R) \simeq \bigoplus_{\alpha_1,\ldots,\alpha_n<0} K \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ as vector spaces, where the action comes from

$$H^n_{(\underline{x})}(R) = \frac{R_{x_1\cdots x_n}}{\sum_i R_{x_1\cdots\widehat{x_i}\cdots x_n}} \simeq \frac{\bigoplus_{\alpha_i\in\mathbb{Z}} K \cdot \underline{x}^{\underline{\alpha}}}{\bigoplus_{\text{some } \alpha_i \geq 0} K \cdot \underline{x}^{\underline{\alpha}}}.$$

Thus,

$$\overline{x}_i \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \begin{cases} x_1^{\alpha_1} \cdots x_i^{\alpha_i+1} \cdots x_n^{\alpha_n} & \text{if } \alpha_i < -1 \\ 0 & \text{if } \alpha_i = -1 \end{cases},$$

and $\partial/\partial x_i \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \alpha_i x_1^{\alpha_1} \cdots x_i^{\alpha_i-1} \cdots x_n^{\alpha_n}$. Moreover, $\partial^{(\beta)}\underline{x}^{\underline{\alpha}} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \cdots \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \underline{x}^{\underline{\alpha}-\underline{\beta}}$.

From these computations, we have that $H^n_{(\underline{x})}(R)$ is generated by $\mathfrak{n} = x_1^{-1} \cdots x_n^{-1}$ as a D-module, and $(0 :_{\mathrm{D}_{R/A}} \mathfrak{n})$ is the set $\mathrm{D}_{R/K} \cdot (x_1, \ldots, x_n)$, a left ideal of $\mathrm{D}_{R/K}$. Thus, $H^n_{(\underline{x})}(R) \simeq\simeq \mathrm{D}_{R/K} / \mathrm{D}_{R/K} \cdot (x_1, \ldots, x_n)$ as D-modules. Likewise, check that $R \simeq \mathrm{D}_{R/K} / \mathrm{D}_{R/K}(\{\partial^{(\alpha)} \mid \alpha \neq 0\})$ as D-modules.

5.3 D-modules and differential equations

To any differential operator $\delta \in \mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}$ there is a differential equation

$$\delta(f) = 0 \tag{5.3.0.1}$$

$$(\overline{\lambda} - \partial/\partial x)(f) = 0$$
$$f = ce^{\lambda x}, \ 1 - \dim \ \text{v.s.} /\mathbb{R}.$$

Likewise one can consider a linear system of PDE'^{s}

$$\begin{bmatrix} \delta_{11} & \cdots & \delta_{1m} \\ \vdots & & \vdots \\ \delta_{n1} & \cdots & \delta_{nm} \end{bmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5.3.0.2}$$

We can express solving 5.3.0.1 or 5.3.0.2 as something algebraic. Generally, people look for solution in

$$\begin{cases} \mathcal{C}^\infty(\mathbb{R}^n) \\ \mathbb{R}[[x_1,\dots,x_n]] \\ \mathbb{R}\{x_1,\dots,x_n\} \end{cases}.$$

Remark 5.3.1. Each of theses is a D-module.

Proposition 5.3.2. *For each of* $M = \begin{cases} \mathcal{C}^\infty(\mathbb{R}^n) \\ \mathbb{R}[[x_1,\dots,x_n]] \\ \mathbb{R}\{x_1,\dots,x_n\} \end{cases}$ *, there is a bijection*

$$\mathrm{Hom}_{\mathbb{R}[\underline{x}]/\mathbb{R}}(\mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}/\mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}\cdot\delta, M) \longrightarrow \{\text{solutions of } \delta(f)=0 \text{ in } M\}.$$

Proof. Any $\mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}$-linear map $\frac{\mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}}{\mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}\cdot\delta} \xrightarrow{\delta} M$ is determined by the image of $\overline{1}$. Moreover, must have

$$0=\sigma(\delta)=\sigma(\delta\overline{1})=\delta\cdot\sigma(\overline{1}),$$

so $\sigma(\overline{1})$ must be a solution of $\delta(\sigma(\overline{1}))=0$. Conversely, if $\delta(f)=0$, there is a map σ with $\sigma(\overline{1})=f$. □

Proposition 5.3.3. *For each M as in Proposition 5.3.2, there exists a bijection*

$$\mathrm{Hom}_{\mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}}(\mathrm{Coker}(\mathrm{D}^a_{\mathbb{R}[\underline{x}]/\mathbb{R}} \xrightarrow{A} \mathrm{D}^b_{\mathbb{R}[\underline{x}]/\mathbb{R}}), M) \longrightarrow \left\{\text{solutions } (f_1,\dots,f_b) \text{ of } A\cdot\begin{pmatrix} f_1 \\ \vdots \\ f_b\end{pmatrix} = \underline{0} \text{ in } M\right\}.$$

Thus, every finitely presented D-module can be though of as linear system of PDE'^{s}

$$\mathbb{R}[\underline{x}] = \frac{\mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}}{\mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}\cdot\{\partial/\partial x_1,\dots,\partial/\partial x_n\}} \longleftrightarrow \frac{\partial}{\partial x_1}(f)=\cdots=\frac{\partial}{\partial x_n}(f)=0.$$

$$H^i_{(\underline{x})}(\mathbb{R}[\underline{x}]) = \frac{\mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}}{\mathrm{D}_{\mathbb{R}[\underline{x}]/\mathbb{R}}\cdot\{\overline{x_1},\dots,\overline{x_n}\}} \longleftrightarrow \overline{x_1}(f)=\cdots=\overline{x_n}(f)=0.$$

$$H^i_{(x_1,\dots,x_n)}(A[x_1,\dots,x_n]) = \begin{cases} 0 & i \text{ if } \neq n \\ \sum_{\alpha_1,\dots,\alpha_n<0} A\cdot x_1^{\alpha_1}\cdots x_n^{\alpha_n} \text{ if } i=n \end{cases}$$

5.4 D-ideals and D-modules simplicity

Definition 5.4.1. A commutative ring R is D-**module simple** (over $A \longrightarrow R$) if it is a simple D-module. This means that the only D-ideals are R and (0).

Remark 5.4.2. Many source say "D-simple" for what we call "D-module simple".

Lemma 5.4.3. *If for every $r \in R \setminus \{0\}$, there is some $\delta \in \mathrm{D}_{R/A}$ with $\delta(r) = 1$, then R is* D*-module simple. The converse holds too.*

Proof. ($\Rightarrow$) Let J be a D-module and $r \in J \setminus \{0\}$ (if $J \neq 0$). Then, there is $\delta \in \mathrm{D}_{R/A}$ with $\delta(r) = 1$. Since J is a D-ideal, $1 \in J$, so $J = R$.

($\Leftarrow$) Observe that $\mathrm{D}_{R/A}(r) \subseteq R$ is a D-ideal. If $r \neq 0$, then this must be all of R, which means that there exists δ with $\delta(r) = 1$. □

Proposition 5.4.4. *If R is Noetherian and* D*-module simple (for any A) then, the zero ideal is primary: every zerodivisor is nilpotent. In particular, if R is reduced, it is a domain.*

Proof. Inmediate from Lemma 5.1.3. □

Example 5.4.5. Let R be a polynomial ring over a field K of characteristic zero. Then, R is D-module simple. Indeed, we proved the much stronger statement

$$\mathrm{D}^i_{R/K} \xrightarrow{\text{res}} \mathrm{Hom}_K([R]_{\leq i}, R)$$

is bijective. One can modify that argument to work in positive characteristic (Exercise: double-check that argument works clear free).

Example 5.4.6. A Stanley-Reisner ring (other than a polynomial ring) is not D-module simple. E.g., for $R = \frac{K[x,y]}{(xy)}$, $(0), (x), (y), (xy)$ are all D-ideals.

Example 5.4.7. The cubic $R = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)$ has no operators of negative degree. Thus, each ideal $[R]_{\geq i}$ is D-ideal, so R is not D-module simple.

Our next goal is to show that direct summands of D-module simple rings are again D-module simple. First, observe:

Lemma 5.4.8. *Let $A \longrightarrow R \longrightarrow S$ be commutative rings, and M, N S-modules. Then,*

$$\mathrm{D}^i_{S/R}(M, N) \subseteq \mathrm{D}^i_{S/A}(M, N) \subseteq \mathrm{D}^i_{R/A}(M, N)$$

for each i.

Proof. Exercise. □

Lemma 5.4.9. *Let $A \longrightarrow R \xrightarrow{i} S$ be commutative rings, and let $\beta : S \longrightarrow R$ be R-linear. Then, there is a map*

$$\begin{aligned} \rho : \mathrm{D}^i_{S/A} &\longrightarrow \mathrm{D}^i_{R/A} \\ \delta &\longrightarrow \beta \circ \delta \circ i. \end{aligned}$$

Proof. We have $\mathrm{D}^i_{S/A} \subseteq \mathrm{D}^i_{R/A}(S,S)$, $\beta \in \mathrm{D}^0_{R/A}(S,R)$, $i \in \mathrm{D}^0_{R/A}(R,S)$, so it is clear from what we know about compositions. ☐

Theorem 5.4.10 (Smith)**.** *Let $A \longrightarrow R \longrightarrow S$ be commutative rings, with S D-module simple (over A), and suppose that R is a direct summand of S. Then, R is D-module simple.*

Proof. Let $\beta : S \longrightarrow R$ be the splitting, so $\beta(1) = \beta(i(1)) = 1$. Given $r \in R$, there exists $\delta \in \mathrm{D}_{S/A}$ with $\delta(i(r)) = 1$. Then, $\beta(\delta(i(r))) = 1$, and $\beta \circ \delta \circ i \in \mathrm{D}_{R/A}$. Thus, R is D-module simple. ☐

Corollary 5.4.11. *A direct summand of a polynomial ring over a field is D-module simple. In particular, rings of invariants of linear reductive groups, e.g. finite groups with $|G| \in K^\times$, $(K^\times)^t$ and others, are D-module simple. Also $R = \mathbb{C}[\{\Delta_{ij} \mid i < j\}]$.*

We now want to relate D-module simplicity to some classes of singularities in positive characteristic.

Recall $F : R \longrightarrow R$ such that $F(r) = r^p$ is a ring homomorphism if R is of characteristic $p > 0$. Likewise $F^e : R \longrightarrow R$ such that $F^e(r) = r^{p^e}$ is iterates. We will write ${}^eR$ for R with the R-module structure via restriction of scalars through F^e. That is $s \cdot r = s^{p^e} \in {}^eR$ with $s \in R$ and $r \in {}^eR$. Thus, $F^e : R \longrightarrow {}^eR$ is R-linear.

R is reduced iff F is injective iff F^e is injective for some e (all e). In this case (R reduced), $Frac(R) = W^{-1}R$, W set of nonzerodivisors on R, is a product of fields, and

$$R \hookrightarrow Frac(R) = \prod_{P \in \mathrm{Min}(R)} Frac(R/P) \hookrightarrow \prod_{P \in \mathrm{Min}(R)} \overline{Frac(R/P)} := \overline{Frac(R)}.$$

Then, we have $R^{1/p^e} := \{r \in \overline{Frac(R)} \mid r^{p^e} \in R\}$ is a subring of $\overline{Frac(R)}$, and the R-module structure on R^{1/p^e} via restriction of scalars $R \subseteq R^{1/p^e}$ agrees with R-module ${}^eR$ via $r \to r^{1/p^e}$:=(element s such that $s^{p^e} = r$, $s \in \overline{Frac(R)}$) with $r \in R$ and $r^{p^e} \in R^{1/p^e}$. Also in this case $R \simeq R^{p^e} \subseteq R$ and we can identify $R \xrightarrow{F^e} {}^eR$ with $R^{p^e} \overset{\text{inclusion}}{\longrightarrow} R$.

$$\begin{array}{ccc} R & \xrightarrow{\;1\;} & R^{1/p^e} \\ \sim\downarrow F^e & & \sim\downarrow F^e \\ R^{p^e} & \xrightarrow{\;1\;} & R \end{array}$$

Definition 5.4.12.

- R is F**-finite** if the Frobenius map $F : R \longrightarrow {}^1R$ is module-finite (iff F is algebra finite iff F^e is module-finite for all $e > 0$ iff F^e is module-finite for some $e > 0$ iff F^e is algebra finite).

- R is **F-split** if the Frobenius map $F : R \longrightarrow {}^1R$ splits as a map of R-modules (iff $F^e : R \longrightarrow {}^eR$ splits for all $e > 0$ iff $F^e : R \longrightarrow {}^eR$ splits for some $e > 0$). This implies that R is reduced.

- R is **strongly F-regular** if for every c not in any minimal prime of R, there exists e such that the map

$$R \xrightarrow{cF^e} {}^eR$$
$$r \longrightarrow cr^{p^e}$$

splits a R-modules (this implies that R is F-split).

Remark 5.4.13.

(1) If R is a F-finite regular ring, then R is strongly F-regular.

(2) A ring is strongly F-regular if and only if it is the product of strongly F-regular domains.

Example 5.4.14. Let K be a perfect field of characteristic $p > 0$, and $R = K[x_1, \dots, x_n]$ be a polynomial ring. Then, $R^{p^e} = K[x_1^{p^e}, \dots, x_n^{p^e}]$.

Any polynomial can be written uniquely as a sum

$$\sum_{\alpha_1, \dots, \alpha_n \le p^e - 1} f \cdot x^\alpha,$$

$f \in R^{p^e}$; i.e.,

$$R = \bigoplus_{\alpha_1, \dots, \alpha_n \le p^e - 1} R^{p^e} x^\alpha,$$

so R is a free R^{p^e}-module with basis $\{x^\alpha \mid \alpha_1, \dots, \alpha_n \le p^e - 1\}$.

Moreover, if $r \in R \backslash (x_1^{p^e}, \dots, x_n^{p^e})$, then r is part of a free basis for R as an R^{p^e}-module: this can be seen since the expression of r in the basis $\{x^\alpha\}$ must contain a unit for some coefficient.

Now, if $c \in R \backslash \{0\}$, we must have $c \notin (x_1^{p^e}, \dots, x_n^{p^e})$ for some e. Then, c is part of a free basis for R over R^{p^e}, so there exists an R^{p^e}-linear projection map sending $c \to 1$. This map is a splitting of cF^e, so R is strongly F-regular.

Theorem 5.4.15 (Smith)**.** *Let R be F-finite and F-split. Then R is strongly F-regular if and only if R is a product of* D*-simple domains.*

Proof. By Remark 5.4.13, this reduces to the case of a domain. R is strongly F-regular if only if for every $c \in R \backslash \{0\}$ there exist $e \in \mathbb{N}$ and $\varphi \in \operatorname{Hom}_{R^{p^e}}(R, R^{p^e})$ such that $\varphi(c) = 1$. Then, postcomposition with inclusion $R^{p^e} \subseteq R$ gives $\widetilde{\varphi} \in \operatorname{Hom}_{R^{p^e}}(R, R)$ with $\widetilde{\varphi}(c) = 1$. Conversely, given $\widetilde{\varphi} \in \operatorname{Hom}_{R^{p^e}}(R, R)$ with $\widetilde{\varphi}(c) = 1$, postcomposition with a splitting $\beta : R \longrightarrow R^{p^e}$ yields $\varphi \in \operatorname{Hom}_{R^{p^e}}(R, R^{p^e})$ sending $\varphi(c) = 1$. $\square$

Corollary 5.4.16. *If R is essential of type finite over K perfect of characteristic $p > 0$, and R is strongly F-regular, then, R is D-module simple.*

Remark 5.4.17. In characteristic $p > 0$, direct summand of regular ring $\overset{\Rightarrow}{\not\Leftarrow}$ strongly F-regular.

5.5 D-algebra simplicity

Recall that a (noncommutative) algebra is simple if it admits no proper quotient rings; equivalently it has no two-sided ideals other than (0).

Definition 5.5.1. Given $A \longrightarrow R$ commutative we say R is **D-algebra simple** if $\mathrm{D}_{R/A}$ is a simple A-algebra.

Proposition 5.5.2. *If R is* D*-algebra simple, then every nonzero* D*-module is a faithful R-module.*

Proof. Every annihilator is a 2-sided ideal, so if $M \neq 0$, it must be 0. Then, if $r \in R$ annihilates M then $\bar{r} \in \mathrm{D}^0_{R/A}$ does as well, so $r = 0$. □

Proposition 5.5.3. *If R is* D*-algebra simple, then R is* D*-module simple.*

Proof. If $I \subseteq R$ is a D-ideal, R/I is a D-module, which has annihilator I. □

Proposition 5.5.4. *If R is* D*-algebra simple, then for every ideal $I \subseteq R$ and every i, $H^i_I(R)$ is either zero or a faithful R-module. Likewise for $H^i_I(M)$, M any* D*-module.*

Proof. Follows from Proposition 5.5.3. □

Example 5.5.5. Let $R = \mathbb{C}[x, xy, y^2, y^3] \subseteq \mathbb{C}[x, y] = S$.

$$R = \bigoplus_{\substack{(i,j)\neq(0,1)\\ i,j\geq 0}} \mathbb{C}\cdot x^i y^j \subseteq S = \bigoplus_{\substack{(i,j)\\ i,j\geq 0}} \mathbb{C}\cdot x^i y^j.$$

Claim: R is D-module simple but not D-algebra simple.

(1) R is not D-algebra simple: consider the short exact sequence of R-modules

$$0 \longrightarrow R \longrightarrow S \longrightarrow \mathbb{C}\cdot y \longrightarrow 0$$

where $\mathfrak{m} = (x, xy, y^2, y^3)$ kills $\mathbb{C}\cdot y$ ($\mathbb{C}\cdot y \simeq R/\mathfrak{m}$). Note that $\sqrt{(x, y^2)} = \mathfrak{m}$.

Property of local cohomology: short exact sequences of modules $\rightsquigarrow$ long exact sequences of cohomology

$$\begin{aligned} 0 &\longrightarrow H^0_{(x,y^2)}(R) \longrightarrow H^0_{(x,y^2)}(S) \longrightarrow H^0_{(x,y^2)}(\mathbb{C}\cdot y) \longrightarrow \\ &\longrightarrow H^1_{(x,y^2)}(R) \longrightarrow H^1_{(x,y^2)}(S) \longrightarrow H^1_{(x,y^2)}(\mathbb{C}\cdot y) \longrightarrow \\ &\longrightarrow H^2_{(x,y^2)}(R) \longrightarrow \cdots \end{aligned}$$

Remark: $H^i_{(x,y^2)}(S) \simeq H^i_{(x,y)}(S)$ for each i. Computation claimed early: $H^i_{(x,y)}(S) = 0$ for $i < 2$. (In general, $H^i_J(T) = 0$ for $i < \operatorname{depth}_J(T)$.)

$$\begin{aligned} H^0_{(x,y^2)}(\mathbb{C}\cdot y) &= \operatorname{Ker}(\mathbb{C}\cdot y \longrightarrow (\mathbb{C}\cdot y)_x \oplus (\mathbb{C}\cdot y)_{y^2}) \\ &\simeq \mathbb{C}\cdot y. \end{aligned}$$

Get from LES

$$0 \longrightarrow \mathbb{C}\cdot y \longrightarrow H^1_{(x,y^2)}(R) \longrightarrow 0$$

so $H^1_{(x,y^2)}(R) \simeq \mathbb{C}\cdot y$ is nonzero, not faithful.

(2) R is D-module simple. Observe: $Frac(R) = Frac(S) = \mathbb{C}(x,y)$, and that

$$\begin{aligned} \mathrm{D}_{R/\mathbb{C}} &= \{\delta \in \mathrm{D}_{\mathbb{C}(x,y)/\mathbb{C}} \mid \delta(R) \subseteq R\} \\ &\subseteq \{\delta \in \mathrm{D}_{S/\mathbb{C}} \mid \delta(R) \subseteq R\}. \end{aligned}$$

Note that $\alpha = \overline{1} - \overline{y}\dfrac{\partial}{\partial y} \in \mathrm{D}_{S/\mathbb{C}}$ can be computed as, for $f = \sum_i f_i(x)y^i \in S$, $\alpha(f) = \sum_i (1-i) f_i(x) y^i \subseteq R$. Also, $\alpha(1) = 1$.

Then, given $f \in R\backslash\{0\}$, there exists $\delta \in \mathrm{D}_{S/\mathbb{C}}$ with $\delta(f) = 1$, since S is D-module simple. Then, $(\alpha \circ \delta) \in \mathrm{D}_{R/\mathbb{C}}$ (since $\alpha(S) \subseteq R$), and $(\alpha\circ\delta)(f) = 1$, so R is D-module simple.

Now our goal is to show that polynomial rings are D-algebra simple.

We will give different proofs in characteristic zero and in positive characteristic $p > 0$.

For characteristic zero, recall that $[\overline{x}^\alpha \partial^{(\beta)}, \overline{x}_i] = \overline{x}^\alpha \partial^{(\beta - e_i)}$ and $[\overline{x}^\alpha\partial^{(\beta)}, \partial^{(e_i)}] = \overline{x}^{\alpha - e_i}\partial^{(\beta)}$, for e_i the i-th standard basis vector.

Theorem 5.5.6. *Let K be a field of characteristic zero. Then, the polynomial ring $R = K[\underline{x}]$ is* D*-algebra simple.*

Proof. Let $J \neq 0$ be a two-sided ideal, and take $\delta \in J$ nonzero. For any $\gamma \in \mathrm{D}_{R/K}$, we have $[\delta, \gamma] \in J$ as well.

Write $\delta = \sum_i \lambda_i \overline{x}^{\alpha_i}\partial^{(\beta_i)}$ some constants $\lambda_i \in K$. Supposing $|\beta_1| \leq |\beta_i|$ all i, we can apply $[-, \overline{x}_j]$ $\beta_{1,j}$-times for each j, and we then get $\overline{r} \in J$. Similarly, taking a largest degree monomial, we can take $[-, \partial^{(e_i)}]$ repeatedly and get $1 \in J$. $\square$

Matrix rings: If R is a commutative ring, and F is a free module of rank n, then a choice of basis for F (i.e. an iso $F \simeq R^{\oplus n}$) induces a isomorphism $\mathrm{End}_R(F) \simeq \mathrm{Mat}_{n\times n}(R)$.

"Left multiplication" in $\mathrm{Mat}_{n\times n}(R) \leftrightsquigarrow$ row operations.

"Right multiplication" in $\mathrm{Mat}_{n\times n}(R) \leftrightsquigarrow$ column operations.

Given a matrix M with a nonzero entry r in any position, can generate (as a two-sided ideal) all matrices with entries in (r). Likewise, if the entries of M generate $I \subseteq R$, then M generates (as a two-sided ideals) $\mathrm{Mat}_{n\times n}(I) \subseteq \mathrm{Mat}_{n\times n}(R)$. All two-sided ideals arise this way .

In particular:

(1) $\mathrm{Mat}_{n\times n}(R)$ is generally not simple (unless R is a field).

(2) A matrix $M \in \mathrm{Mat}_{n\times n}(R)$ generates the whole matrix ring as a two-side ideals if it has a unit entry.

An element $\varphi \in \mathrm{End}_R(F)$ (F free module) generates the whole endo. ring if it has a unit entry with respect to any free basis for F.

Theorem 5.5.7. *Let K be a perfect field of characteristic p. Then, the polynomial ring $R = K[\underline{x}]$ is* D*-algebra simple.*

Proof. Let $J \neq 0$ be a two-sided ideal, and take $\delta \in J$ nonzero. We have $\delta \in \mathrm{Hom}_{R^{p^e}}(R, R)$ for some e. Since R is a free R^{p^a}-module of finite rank, $\mathrm{Hom}_{R^{p^a}}(R, R)$ is a matrix ring for each a.

Thus, if δ considered as an element of $\mathrm{Hom}_{R^{p^a}}(R, R)$ for some $a \geq e$ has a unit as an entry thought of as a matrix, then $J = R$, as required.

Consider $\delta \in \mathrm{Hom}_{R^{p^e}}(R, R)$ as a matrix with entries in R^{p^e}; let r^{p^e} be a nonzero entry. We have that r is part of a free basis for R as an R^{p^c}-module for some c.

Thus, $\overline{r} \in \mathrm{Hom}_{R^{p^c}}(R, R)$ has 1 as an entry in some free basis for R over R^{p^e}: this is the entry with column corresponding to 1 and row corresponding to r in the basis.

Likewise, $\overline{r}^{p^e} \in \mathrm{Hom}_{R^{p^{e+c}}}(R, R)$ has 1 as an entry in some basis. We can then change a free basis for R over $R^{p^{e+c}}$ in which $\delta \in \mathrm{Hom}_{R^{p^{e+c}}}(R, R)$ has a unit entry, and we are done. □

Corollary 5.5.8. *Let R be a polynomial ring over a perfect field. Then, every local cohomology module on R is faithful as R-module.*

Exercise 5.5.9. *The perfect field hypothesis can be removed, e.g., by a faithfully flat base change argument.*

We now want to show that D-algebra simplicity implies Cohen-Macaulayness.

Definition 5.5.10. A local ring $(R, \mathfrak{m})$ is **Cohen-Macaulay** (CM) if $\mathrm{depth}_{\mathfrak{m}}(R) = \dim(R)$. A ring R is CM if $R_{\mathfrak{p}}$ is CM for every $\mathfrak{p} \in \mathrm{Spec}(R)$.

Facts:

(1) The ring definition does not contradict the local definition: $(R, \mathfrak{m})$ local $\operatorname{depth}_{\mathfrak{m}}(R) = \dim(R) \Rightarrow \operatorname{depth}_{\mathfrak{p}}(R_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

(2) $(R, \mathfrak{m})$ is CM $\Leftrightarrow H_{\mathfrak{m}}^{<\dim(R)}(R) = 0 \Leftrightarrow H_{\mathfrak{m}}^{<\operatorname{ht}(\mathfrak{p})}(R_{\mathfrak{p}}) = 0$ all prime $\mathfrak{p}$.

(3) If $(R, \mathfrak{m})$ is local essential of finite type over a field K and $R_{\mathfrak{p}}$ is CM for all $\mathfrak{p} \neq \mathfrak{m}$, then $H_{\mathfrak{m}}^i(R)$ has finite length as R-module for $i < \dim(R)$.

Theorem 5.5.11 (Van den Bergh)**.** *Let R be essential of finite type over a field K, and suppose that R is* D*-algebra simple. Then, R is CM.*

Proof. If not, pick some $\mathfrak{p} \in \operatorname{Spec}(R)$ with $R_{\mathfrak{p}}$ not CM, but $R_{\mathfrak{q}}$ CM for all $\mathfrak{q} \subsetneq \mathfrak{p}$ (can do by Fact 1). By the previous fact (Fact 3), we have that for some $i < \operatorname{ht}(\mathfrak{p}) = H^i_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ has finite length as an $R_{\mathfrak{p}}$-module, and is nonzero by hypothesis (and Fact 2). Then, $\mathfrak{p}^n H^i_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$ for some $i < \operatorname{ht}(\mathfrak{p})$. But this is $\mathfrak{p}^n H^i_{\mathfrak{p}}(R_{\mathfrak{p}}) = 0$, and since $R_{\mathfrak{p}}$ is a D-module, $H^i_{\mathfrak{p}}(R_{\mathfrak{p}})$ must be faithful, which is a contradiction. □

Recall: D-algebra simple $\Rightarrow$ CM.
D-algebra simple $\overset{\Rightarrow}{\not\Leftarrow}$ D-module simple.
Classical invariant rings $\Rightarrow$ direct summands of polynomial rings $\Rightarrow$ D-module simple.

Conjecture 5.5.12. Classical invariant rings (in characteristic zero) are D-algebra simple.

Many cases are known (LS, Schwarz), but this is an open question still.
We will do finite group invariants biter. The characteristic p analogue of LS conjecture was settled by Smith-Vanden Bergh.

Example 5.5.13. Let $R = K[x, xy, y^2, y^3]$, and $\mathfrak{m} = (x, xy, y^2, y^3)$. Then, $R_{\mathfrak{m}}$ is not CM but is CM on spectrum. Is a 2-dim domain $\mathfrak{p} \subsetneq \mathfrak{m}$, then $R_{\mathfrak{p}}$ is a domain of dim ≤ 1.

Chapter 6

Filtration and Noetherianty

Recall: $(T, F^\bullet)$ is a **filtered ring** if T is a ring with $F^0 \subseteq F^1 \subseteq F^2 \subseteq \cdots$ abelian groups such that:

- $\bigcup_i F^i = T$ (exhaustive).
- $F^i F^j \subset F^{i+j}$ (multiplicative).

If T is on A-algebra, $(T, F^\bullet)$ is a **filtered A-algebra** is also $A \subseteq F^0$ if only if each F^i is an A-module.

If M is a left (right) T-module and $(T, F^\bullet)$ is a filtered ring, then $G^\bullet$ is a filtration on M consistent with $F^\bullet$ or $(M, G^\bullet)$ is a filtered left (right) $(T, F^\bullet)$-module if $F^i G^j \subseteq G^{i+j}$ (right module $G^j F^i \subseteq G^{i+j}$).

If $(T, F^\bullet)$ is a filtered A-algebra, then $gr(T, F^\bullet) = \bigoplus_i F^i/F^{i-1}$ is a graded A-algebra with $A \subseteq gr(T, F^\bullet)_0$.

If $(M, G^\bullet)$ is a filtered left (right) $(T, F^\bullet)$-module, then $gr(M, G^\bullet) = \bigoplus_i G^i/G^{i-1}$ is a graded left (right) $gr(T, F^\bullet)$-module.

Example 6.0.1. Let K be afield of characteristic zero, and $R = K[\underline{x}]$ be a polynomial ring. Then, $(\mathrm{D}_{R/K}, \mathrm{D}^\bullet_{R/K})$ (order filtration) is a filtered K-algebra, and

$$\begin{aligned} gr^{\mathrm{ord}}(\mathrm{D}_{R/K}) &:= gr(\mathrm{D}_{R/K}, \mathrm{D}^\bullet_{R/K}) \\ &= K[y_1, \ldots, y_n, z_1, \ldots, z_n] \end{aligned}$$

polynomial ring with $y_i = \overline{x}_i$ degree 0, $z_i = \frac{\partial}{\partial x_i} + \mathrm{D}^0_{R/K}$, degree 1.

More generally, write $gr^{\mathrm{ord}}(\mathrm{D}_{R/K})$ for $gr(\mathrm{D}_{R/K}, \mathrm{D}^\bullet_{R/K})$.

Recall that $gr^{\mathrm{ord}}(\mathrm{D}_{R/K})$ is always commutative.

Lemma 6.0.2. *Let K be a field of characteristic zero, and G be a finite group. Then, the functor*

$$(-)^G : K[G]-\mathrm{mod} \longrightarrow K-\mathrm{mod}$$

is exact.

Proof. In general (no assumption on characteristic), the invariants functor is left-exact (exercise).

To see exactness, need to check it preserves surjections. Given a $K[G]$-module M, there is a projection map

$$\mathcal{P}_M : M \longrightarrow M^G$$
$$\mathcal{P}_M(m) = \frac{1}{|G|}\sum_{g\in G} g\cdot m.$$

For a $K[G]$-linear map $M \xrightarrow{\alpha} N$, have commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ \downarrow{\scriptstyle \mathcal{P}_M} & & \downarrow{\scriptstyle \mathcal{P}_N} \\ M^G & \xrightarrow{\alpha|_{M^G}} & N^G \end{array}.$$

Thus, if α is surjective, $\alpha|_{M^G}$ is surjective as well. □

Example 6.0.3. Let K be a field of characteristic zero. Let $R = [\underline{x}]$, and G be a finite group acting linearly on R with no pseudorefletions. Then, G acts on $\mathrm{D}\, R/K$ by conjugation and preserves the order filtration: $g(\mathrm{D}\, R/K^i) \subseteq \mathrm{D}\, R/K^i$ for each i.

We have $\mathrm{D}\, R^G/K^i \simeq (\mathrm{D}\, R/K^i)^G$ by Kantor's Theorem. Since G preserves order filtration, G acts on $gr^{\mathrm{ord}}(\mathrm{D}\, R/K)$ by $g\cdot(\delta + \mathrm{D}\, R/K^{i-1}) = (g\cdot\delta + \mathrm{D}\, R/K^{i-1}$ so,

$$0 \longrightarrow \mathrm{D}^{i-1}_{R/K} \longrightarrow \mathrm{D}^{i}_{R/K} \longrightarrow gr^{\mathrm{ord}}(\mathrm{D}_{R/K})_i \longrightarrow 0$$

as $K[G]$-modules. Then,

$$\begin{aligned} gr^{\mathrm{ord}}(\mathrm{D}_{R^G/K})_i &\simeq \frac{\mathrm{D}^i_{R^G/K}}{\mathrm{D}^{i-1}_{R^G/K}} \\ &\simeq \frac{(\mathrm{D}^i_{R/K})^G}{(\mathrm{D}^{i-1}_{R/K})^G} \\ &\simeq \left(\frac{\mathrm{D}^i_{R/K}}{\mathrm{D}^{i-1}_{R/K}}\right)^G \\ &\simeq gr^{\mathrm{ord}}(\mathrm{D}_{R/K})^G_i \end{aligned}$$

so, $gr^{\mathrm{ord}}(\mathrm{D}_{R^G/K}) \simeq gr^{\mathrm{ord}}(\mathrm{D}_{R/K})^G$.

By Noether's finiteness theorem for invariants (on polynmial rings), $gr^{\mathrm{ord}}(\mathrm{D}_{R^G/K})$ is finitely generated K-algebra, hence Noeth, and

$$gr^{\mathrm{ord}}(\mathrm{D}_{R^G/K}) \hookrightarrow gr^{\mathrm{ord}}(\mathrm{D}_{R/K})$$

is module finite.

Exercise 6.0.4. *Let $R = \mathbb{C}[y] \to S = \mathbb{C}[x]$ via $y \to x^2$. Then, $R = S^G$ where $G = \{1, g\}$ with $g \cdot x = -x$, and the operator $\frac{\partial}{\partial y} \in \mathrm{D}_{R/\mathbb{C}}$ does not extend to a differential operator on S ($\Leftrightarrow$ "no pseudoreflections" is necessary in Kantor's Theorem).*

Proposition 6.0.5. *Let $(T, F^\bullet)$ be a filtrated ring and $(M, G^\bullet)$ be a filtered left/right $(T, F^\bullet)$-module. Let $m_1, \dots, m_t \in M$ be such that $m_1 + G^{1-1}, \dots, m_t + G^{t-1} \in gr(M, G^\bullet)$ form a generated set as a left/right $gr(T, F^\bullet)$-module. Then, $m_1, \dots, m_t$ form a generated set as a left/right T-module.*

Proof. By hypothesis, we have $gr(M, G^\bullet)_n = \sum_i gr(T, F^\bullet)_n - d_i(m_i + G^{i-1})$ so, $G_n/G_{n-1} = \sum_i F_{n-d_i}/F_{n-d_i-1} \cdot (m_i + G^{i-1})$, so $G_n = (\sum_i F_{n-d_i} \cdot m_i) + G_{n-1}$ for each n. Thus, $G_n \subseteq \sum_i T_i + G_{n-1}$ for each n. Then, for $n = 0$, $G_{-1} = 0$, so $G_0 \subseteq \sum_i T_i$, and if $G_{n-1} \subseteq \sum_i T_i$, then $G_n \subseteq \sum_i T_i$, so by induction on n, the $m_i's$ generate. □

Proposition 6.0.6. *A ting T is **left Noetherian** if the following equivalent conditions hold:*

i) any ascending chain of left ideals stabilizes,

ii) every nonempty family of left ideals has a maximal element,

iii) every left ideal id finitely generated,

iv) every left submodule of a finitely generated left module is finitely generated,

v) every finitely generated left module is finitely presented.

Proof. Exercise (similar to commutative case). □

Proposition 6.0.7. *If $(T, F^\bullet)$ is a filtered ring and $gr(T, F^\bullet)$ is left (right) Noetherian, then T is left (right) Noetherian.*

Proof. If $J \subseteq T$ is a left ideal, then $(J, J \cap F^\bullet)$ is a filtered left $(T, F^\bullet)$-module.

In this case

$$gr(J, J \cap F^\bullet) \hookrightarrow gr(T, F^\bullet),$$

so this identifies with a left ideal, which by hypothesis is finitely generated.

Then, by Proposition 6.0.6, J is finitely generated, so T is left Noetherian. □

Theorem 6.0.8. *Let K be a field of characteristic zero, R polynomial ring over K. Then,* $\mathrm{D}_{R/K}$ *is left and right Noetherian.*

Theorem 6.0.9. *Let K be a field of characteristic zero, R polynomial ring over K. If G is finite, acts linearly on R with no pseudoreflections, then* $\mathrm{D}_{R^G/K}$ *is left and right Noetherian, and* $\mathrm{D}_{R/K}$ *is finitely presented left/right* $\mathrm{D}_{R^G/K}$*-module.*

Remark 6.0.10. It is not true that $\mathrm{D}_{R/K}$ is left Noetherian $\Rightarrow gr^{\text{ord}}(\mathrm{D}_{R/K})$ Noetherean.

Example 6.0.11. For $R = \mathbb{C}[x, y]/(xy)$

i) $gr^{\text{ord}}(\mathrm{D}_{R/\mathbb{C}})$ is not Noetherean,

ii) $\mathrm{D}_{R/\mathbb{C}}$ is both left and right Noetherian.

Example 6.0.12. For all SR rings R $\mathrm{D}_{R/K}$ is right Noetherian, but some, not all are left Noetherian e.g, $R = \frac{\mathbb{C}[x,y,u,v]}{(xu,xv,yu,yv)}$, $\mathrm{D}_{R/\mathbb{C}}$ is not left Noetherian.

6.1 Polynomial rings in characteristic $p > 0$

Lemma 6.1.1 (Lucas' Theorem)**.** *Let p be prime, and $m = \sum_{i=0}^k m_i p^i$, $n = \sum_{i=0}^k n_i p^i$, $0 \le m_i, n_i < p$ (base p-expansion). Then, $\binom{m}{n} \equiv \prod_{i=0}^k \binom{m_i}{n_i}$ mod p. (we take $\binom{m}{n} = 0$ for $m < n$).*

Proof. In $\mathbb{F}_p[x]$, we have

$$\begin{aligned}
\sum_{n=0}^{m} \binom{m}{n} x^n &= (1+x)^m \\
&= (1+x)^{\sum_{i=0}^k m_i p^i} \\
&= \prod_{i=0}^{k} ((1+x)^{p^i})^{m_i} \\
&= \prod_{i=0}^{k} (1+x^{p^i})^{m_i} \\
&= \prod_{i=0}^{k} \sum_{n_i=0}^{m_i} \binom{m_i}{n_i} x^{n_i p^i} \\
&= \prod_{i=0}^{k} \sum_{n_i=0}^{p-1} \binom{m_i}{n_i} x^{n_i p^i} \\
&= \sum_{n=0}^{m} \left(\prod_{i=0}^{k} \binom{m_i}{n_i} \right) x^n.
\end{aligned}$$

□

Corollary 6.1.2. *If there exists e such that $a, b < p^e$, $a + b \geq p^e$, then $\binom{a+b}{a} \equiv 0$ mod p.*

Proof. We claim that there is a base p digit of $a+b$ that is small then the corresponding digit of a. Otherwise, setting $c = a + b$,

$$c = c_e p^e + c_{e-1}p^{e-1} + \cdots + c_0$$
$$a = a_{e-1}p^{e-1} + \cdots + a_0$$

$c_e > 0,\ 0 \leq a_i \leq c_i < p$ for $i < e$.

$$b = b_e p^e + b_{e-1}p^{e-1} + \cdots + b_0$$

$b_i = c_i - a_i$ for all i is base p expansion. This contradicts that $b < p^e$. Done by Theorem 6.1.1. □

Notation 6.1.3. For n-netuples α, β we will write $\alpha! = \alpha_1! \cdots \alpha_n!$ and $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n} = \frac{\beta!}{\alpha!(\beta-\alpha)!}$.

Lemma 6.1.4. *Let $R = A[\underline{x}]$ be a polynomial ring. Then, $\partial^{(\alpha)}\partial^{(\beta)} = \binom{\alpha+\beta}{\alpha}\partial^{(\alpha+\beta)}$.*

Proof. Evaluate at x^σ

$$\begin{aligned}\partial^{(\alpha)}\partial^{(\beta)}(x^\sigma) &= \partial^{(\alpha)}\left(\binom{\sigma}{\beta}x^{\sigma-\beta}\right)\\ &= \binom{\sigma-\beta}{\alpha}\binom{\sigma}{\beta}x^{\sigma-(\alpha+\beta)}\\ &= \frac{(\sigma-\beta)!\sigma!}{(\sigma-(\alpha+\beta))!\alpha!(\sigma-\beta)!\beta!}x^{\sigma-(\alpha+\beta)}.\end{aligned}$$

$$\begin{aligned}\partial^{(\alpha+\beta)}(x^\sigma) &= \binom{\sigma}{\alpha+\beta}x^{\sigma-(\alpha+\beta)}\\ &= \frac{\sigma!}{(\sigma-(\alpha+\beta))!(\alpha+\beta)!}x^{\sigma-(\alpha+\beta)}\end{aligned}$$

$$\binom{\alpha+\beta}{\alpha}\partial^{(\alpha+\beta)}(x^\sigma) = \frac{(\alpha+\beta)!\sigma!}{\alpha!\beta!(\sigma-(\alpha+\beta))!(\alpha!+\beta)!}x^{\sigma-(\alpha+\beta)}.$$

Since the operator agree on A-module generator set for R and they are A-linear, they agree as functions (operators) on R. □

Theorem 6.1.5. *Let K be a field of characteristic $p > 0$, and $R = [\underline{x}]$ be a polynomial ring. Then, $\mathrm{D}_{R/K}$ is not left Noetherian.*

Proof. Let $J_e = \mathrm{D}_{R/K} \cdot \{\partial^{(\alpha)} \mid \alpha_1 > 0, \alpha_i < p^e \text{ for each } i\}$. These from an ascending chain of left ideals. We have $\partial^{(p^e,0,\dots,0)} \notin J_a,\ a < e$. Write $\mathrm{D}_{R/K} = \bigoplus \overline{R}\partial^{(\beta)}$. Then,

$$\begin{aligned}\mathrm{D}_{R/K} \cdot \{\partial^{(\alpha)} \mid \alpha_1 > 0, \alpha_i < p^e\} &= \sum \overline{R}\partial^{(\beta)}\partial^{(\alpha)} \\ &= \sum \binom{\alpha+\beta}{\alpha} \overline{R}\partial^{(\alpha+\beta)}.\end{aligned}$$

This implies proper ascending infinite chain of left ideals. □

Example 6.1.6. Let $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$. $\mathrm{D}_{R/\mathbb{C}}$ is not left Noetherian, even worse, there is an infinite ascending proper chain of two-sided ideals.

$$J_k = \mathbb{C}[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}][\mathrm{D}^k]_1 + [\mathrm{D}]_{\geq 2}$$

is a two-sided ideal since $[\mathrm{D}]_{<0} = 0$, $[\mathrm{D}]_0 = \mathbb{C}[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}]$. Is a proper chain by earlier claim.

Next goal: For K a field of characteristic zero, $R = K[x,y]/(xy)$,

- $gr^{\mathrm{ord}}(\mathrm{D}_{R/K})$ is not Noetherian.
- $\mathrm{D}_{R/K}$ is not left and right Noetheria.

Definition 6.1.7. Given two left/right $(T, F^\bullet)$-modules, $(M, G^\bullet)$, $(N, H^\bullet)$ we say a T-module homomorphism $M \xrightarrow{\varphi} N$ is a map of filtered modules or a $(T, F^\bullet)$-module homomorphism if $\varphi(G^i) \subseteq H^i$ for all i.

Exercise 6.1.8. *If $(L, E^\bullet)$, $(M, G^\bullet)$, $(N, H^\bullet)$ are $(T, F^\bullet)$-modules (left or right), and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a SES, where each map is a map of $(T, F^\bullet)$-modules, then

$$0 \longrightarrow gr(L, E^\bullet) \longrightarrow gr(M, G^\bullet) \longrightarrow gr(N, H^\bullet) \longrightarrow 0$$

is a SES of $gr(T, F^\bullet)$-modules.

Recall that if $S = A[\underline{x}]$ a polynomial ring, and $R = S/I$, then

$$\mathrm{D}^i_{R/A} \simeq \frac{(I :_{\mathrm{D}^i_{S/A}} I)}{I\,\mathrm{D}^i_{S/A}} \simeq \frac{(I :_{\mathrm{D}^i_{S/A}} I)}{(I :_{\mathrm{D}^i_{S/A}} S)}.$$

Then,

$$\mathrm{D}_{R/A} \simeq \frac{(I :_{\mathrm{D}_{S/A}} I)}{(I :_{\mathrm{D}_{S/A}} S)}.$$

$(I :_{\mathrm{D}_{S/A}} I)$ is a subring of $\mathrm{D}_{S/A}$ and $(I :_{\mathrm{D}_{S/A}} S)$ is a two-sided ideal of $(I :_{\mathrm{D}_{S/A}} I)$.

This preserves the order filtration, so by Exercise 6.1.8, get SES

$$0 \longrightarrow I gr^{\mathrm{ord}}(\mathrm{D}_{S/A}) \longrightarrow gr^{\mathrm{ord}}(I :_{\mathrm{D}_{S/A}} I) \longrightarrow gr^{\mathrm{ord}}(\mathrm{D}_{R/A}) \longrightarrow 0$$

and $gr^{\mathrm{ord}}(I :_{\mathrm{D}_{S/A}} I) \hookrightarrow gr^{\mathrm{ord}}(\mathrm{D}_{S/A})$.

Back to $R = K[x,y]/(xy)$, we computed eartier that

$$((xy :_{\mathrm{D}_{K[x,y]/K}} (xy)) = K \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} \overline{x}^i \frac{\partial_j}{\partial x} \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} \overline{y}^i \frac{\partial_j}{\partial y} \oplus \bigoplus_{i,j>0} \overline{x}^i \overline{y}^j \frac{\partial}{\partial x}^a \frac{\partial}{\partial y}^b$$

$$gr^{\mathrm{ord}}((xy : (xy)) \hookrightarrow gr^{\mathrm{ord}}(\mathrm{D}_{K[x,y]/K}) \simeq K[x,y,u,v]$$

with $x = [\overline{x}]$, $y = [\overline{y}]$, $v = \left[\frac{\partial}{\partial y}\right] + \mathrm{D}^0$, $u = \left[\frac{\partial}{\partial x}\right] + \mathrm{D}^0$.

Have,

$$gr^{\mathrm{ord}}((xy) : (xy)) \simeq K \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} x^i u^j \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} y^i v^j \oplus \bigoplus_{i,j>0} x^i y^j u^a v^b \subseteq K[x,y,u,v].$$

$$\begin{aligned} gr^{\mathrm{ord}}(\mathrm{D}_{R/K}) &\simeq \frac{gr^{\mathrm{ord}}((xy : (xy))}{(xy)} \\ &\simeq K \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} x^i u^j \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} y^i v^j \\ &\simeq K[x, xu, xu^2, xu^3, \ldots, y, yv, yv^2, yv^3, \ldots] \\ &\subseteq \frac{K[x,y,u,v]}{(xy)} \end{aligned}$$

$K[x, xu, xu^2, xu^3, \ldots, y, yv, yv^2, yv^3, \ldots]$ is nonNoetherian commutative ring quotient ring. $K[x, xu, xu^2, xu^3, \ldots] \subseteq K[x,u]$.

Lemma 6.1.9. *$((x) : (x))$ is right Noetherian.*

Proof. Call $A = ((x) : (x))$, which is a subring of $D = \mathrm{D}_{K[x]/K}$. Note that $D = A \oplus \bigoplus_{i>0} K \left(\frac{\partial}{\partial x}^i\right)$, as vector space. Let $J \subseteq A$ be a right ideal, want to see that J is finitely generated.

Since D is right Noetherian, there are finitely many elements $\underline{f} = f_1, \ldots, f_n \in J$ such that $(\underline{f})D = JD$. If $(\underline{f})A = J$, we are done. If $(\underline{f})A \neq J$, pick $\beta \in J \backslash \underline{f})A$. Since $\beta \in J \subseteq J\bar{D} \subseteq \underline{f})D$ can write

$$\beta = \alpha + \gamma_1 \frac{\partial}{\partial x} + \cdots + \gamma_r \left(\frac{\partial}{\partial x}\right)^r$$

with $\alpha \in (\underline{f})A,\ \gamma_i \in (\underline{f}) \cdot K$

Claim: $\gamma_i \frac{\partial}{\partial x} \in (\underline{f}) \frac{\partial}{\partial x} \cdot K \cap J$.

Proof of claim: Just need to see that each is in J. Wil do a trick

$$\gamma_1 + \sum_{i=2}^{r} i\gamma_i \left(\frac{\partial}{\partial x}\right)^{i-1} = \sum_{i=1}^{r} i\gamma_i \left(\frac{\partial}{\partial x}\right)^{i-1} = \sum_{i=1}^{r} \gamma_i \left(\left(\frac{\partial}{\partial x}\right)^i \bar{x} - \bar{x} \left(\frac{\partial}{\partial x}\right)^i\right).$$

But, $\gamma_1 \in J$ and $\sum_{i=1}^{r} \gamma_i \left(\left(\frac{\partial}{\partial x}\right)^i \bar{x} - \bar{x} \left(\frac{\partial}{\partial x}\right)^i\right) \in J$, then $\sum_{i=2}^{r} i\gamma_i \left(\frac{\partial}{\partial x}\right)^{i-1} \in J$.

Repeat: $\sum_{i=2}^{r} i(i-1)\gamma_i \left(\frac{\partial}{\partial x}\right)^{i-2} = \sum_{i=2}^{r} i\gamma_i \left(\left(\frac{\partial}{\partial x}\right)^{i-1} \bar{x} - \bar{x} \left(\frac{\partial}{\partial x}\right)^{i-1}\right)$ implies

$$\sum_{i=3}^{r} i(i-1)\gamma_i \left(\frac{\partial}{\partial x}\right)^{i-2} \in J \ldots$$

$r!\gamma_r \left(\frac{\partial}{\partial x}\right) \in J$, then $\gamma_r \left(\frac{\partial}{\partial x}\right) \in J$.

Then,

$$\begin{aligned} \gamma_r \left(\frac{\partial}{\partial x}\right)^r &= \gamma_r \left(\frac{\partial}{\partial x}\bar{x} - \bar{x}\frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial x}\right)^r \\ &= -\gamma_r \bar{x} \left(\frac{\partial}{\partial x}\right)^{r+1} + \gamma_r \bar{x} \left(\frac{\partial}{\partial x}\right)^{r+1} \\ &\in J. \end{aligned}$$

Thus,

$$\beta - \gamma_r \left(\frac{\partial}{\partial x}\right)^r = \alpha + \gamma_1 \frac{\partial}{\partial x} + \cdots + \gamma_{r-1} \left(\frac{\partial}{\partial x}\right)^{r-1} \in J.$$

By some argument (decreasing induction on i), get that each $\gamma_i \frac{\partial}{\partial x} in J$.

The, claim implies that J is generated by $(\underline{f})$ and the finite dimension vector space $(\underline{f}) \frac{\partial}{\partial x} \cdot K \cap J$. Put together, get finite generating set for J. □

If T is a noncommutaitve ring, then we can a polynomial ring over T with commutative variables $T[\underline{x}]$. If T is an algebra over a field K, then $T[\underline{x}] \simeq T \otimes_K K[\underline{x}]$.

Exercise 6.1.10. *If T is left/right Noetherian, then $T[\underline{x}]$ is left/right Noetherian. (Hint: Usual proof of Hilbert Basis Theorem.)*

Theorem 6.1.11 (Tripp). *$((xy):(xy))$ is right Noetherian. Hence $\mathrm{D}_{\frac{K[x,y]}{(xy)/K}}$ is right Noetherian.*

Proof. (sketch): Let $S = ((y) :_{\mathrm{D}_{K[y]/K}} (y))$. Then, $((xy):(xy)) \simeq S \otimes_K ((x):(x))$. Call this ring A. Note that A is a subring of $D = S \otimes_K \mathrm{D}_{K[x]/K}$. Proceeded similar to Lemma 6.1.9...

Need to see that D is right Noetherian filter D by $F^i = S \otimes_K \mathrm{D}^i_{K[x]/K}$. Then,

$$\begin{aligned} gr(F^i) &\simeq gr(S \otimes_K \mathrm{D}^\bullet_{K[x]/K}) \\ &\simeq S[z_1, z_2] \end{aligned}$$

polynomial ring over S. Then, right Noetherian by Exercise 6.1.10 implies D is right Noetherian.

Then, some computational trick shows that

$$J = (\underline{f})A + \left((\underline{f})\frac{\partial}{\partial x}(S \otimes 1) \cap J\right) A,$$

this implies J is finitely generated. □

We also want to see left Noetherian. Will use opposite ring to see this.

Definition 6.1.12. The opposite ring of a noncommutative ring T is the ring T^{op}, which as additive groups is identical to T, and has multiplication $\star$

$$r \star s = sr \ (T - \text{multiplication}),$$

will use the convention that $\star$ means "*op*" multiplication and usual multiplication notation means usual T-multiplication.

There is a natural bijection between left T-modules and right T^{op}-modules: if M is a left T-module, then it is a right T^{op}-module by

$$m \cdot t = t \cdot m \ (\text{left } T - \text{action on} M)$$

$m \in M, \ t \in T^{op} (= T)$. Since

$$\begin{aligned} m \cdot (t \star s) &= m \cdot (st) \\ &= (st) \cdot m \\ &= s \cdot (t \cdot m) \\ &= (m \cdot t) \cdot s. \end{aligned}$$

In particular, left ideals of $T \leftrightsquigarrow$ right ideals of T^{op}, so T is left Noetherian iff T^{op} is right Noetherian. Note also, $(T^{op})^{op} = T$ as ring.

Goal: to show $((xy):(xy)) = ((xy):(xy))^{op}$. Want to see this symmetry property for polynomial rings first.

Remark 6.1.13. If $\alpha : T \longrightarrow T^{op}$ homomorphism, then $\alpha : T^{op} \longrightarrow T$ is also a homomorphism. Clear for $+, -$, and

$$\alpha(r \star s) = \alpha(sr) = \alpha(s) \star \alpha(r) = \alpha(r)\alpha(s).$$

A ring ismomorphism $T \longrightarrow T^{op}$ is also called an **antiisomorphism** $T \longrightarrow T$.

Lemma 6.1.14. *Let $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{N}^n$.*

1) $\binom{\alpha+\beta}{\gamma} = \sum_{\delta+\epsilon=\gamma} \binom{\alpha}{\delta}\binom{\beta}{\epsilon}$*;*

2) $\sum_{\beta \leq \alpha} (-1)^{\beta} \binom{\alpha}{\beta} = 0$ *if* $\alpha \neq 0$ *(*$= 1$ *if* $\alpha = 0$*).*

Proof. Exercise (follow from usual binomial coefficient identities). □

Proposition 6.1.15. *Let A be a commutative ring, $R = A[\underline{x}]$ be a polynomial ring. Then,* $\partial^{(\alpha)}\overline{f} = \sum_{\beta+\gamma=\alpha} \overline{\partial^{(\beta)}(f)}\partial^{(\gamma)}$ *in* $\mathrm{D}_{R/A}$.

Proof. First, let $f = x^{\mu}$ monomial. It suffices to check the equality by plugging in x^{σ} to both sides. We have:

$$(\partial^{(\alpha)}\overline{x}^{\mu})(x^{\sigma} = \binom{\mu+\sigma}{\alpha} x^{\mu+\sigma-\alpha}$$

and

$$\begin{aligned}\sum_{\beta+\gamma=\alpha} \overline{\partial^{(\beta)}(x^{\mu})}\partial^{(\gamma)}(x^{\sigma}) &= \sum_{\beta+\gamma=\alpha} \overline{\binom{\mu}{\beta} x^{\mu-\beta}} \binom{\sigma}{\gamma} x^{\sigma-\gamma} \\ &= \sum_{\beta+\gamma=\alpha} \binom{\mu}{\beta}\binom{\sigma}{\gamma} x^{\mu+\sigma-\alpha}\end{aligned}$$

equal by Lemma 6.1.14.

For general $f = \sum_{\sigma} a_{\sigma} x^{\sigma}$,

$$\begin{aligned}\partial^{(\alpha)}\overline{f} &= \partial^{(\alpha)} \cdot \left(\overline{\sum_{\sigma} a_{\sigma} x^{\sigma}}\right) \\ &= \sum_{\sigma} a_{\sigma} \partial^{(\alpha)} \overline{x}^{\sigma} \\ &= \sum_{\sigma} a_{\sigma} \left(\sum_{\beta+\gamma=\alpha} \overline{\partial^{(\beta)}(x^{\alpha})}\partial^{(\gamma)}\right) \\ &= \sum_{\beta+\gamma=\alpha} \overline{\partial^{(\beta)}(f)}\partial^{(\gamma)}.\end{aligned}$$

□

Proposition 6.1.16 (Quinlan-Gallego)**.** *For $R=[\underline{x}]$ a polynomial ring, the map*

$$\psi : \mathrm{D}_{R/A} \longrightarrow \mathrm{D}_{R/A}^{op}$$

given by $\psi(\overline{r}\partial^{(\alpha)}) = (-1)^{|\alpha|}\partial^{(\alpha)}\overline{r}$ is a ring isomorphism. (where $|\alpha| = \alpha_1 + \cdots + \alpha_n$).

Does this specify a unique map? Yes, since $\mathrm{D}_{R/A} = \bigoplus_\alpha \overline{R}\partial^{(\alpha)}$.

Proof. Need to show this is multiplicative. Since any ett. is a sum of elements of the form $\overline{r}\partial^{(\alpha)}$, it suffices to show

$$\psi(\overline{r}\partial^{(\alpha)}\overline{s}\partial^{(\beta)}) = \psi(\overline{r}\partial^{(\alpha)}) \star \psi(\overline{s}\partial^{(\beta)}) = \psi(\overline{r}) \star \psi(\partial^{(\alpha)}) \star \psi(\overline{s}) \star \psi(\partial^{(\beta)}).$$

So, it suffices to show

i) $\psi(\overline{r}\delta) = \psi(\overline{r})\psi(\delta)$ any $r \in R$, $\delta \in \mathrm{D}_{R/A}$;

ii) $\psi(\delta\partial^{(\alpha)}) = \psi(\delta) \star \psi(\partial^{(\alpha)})$ any $\alpha, \delta \in \mathrm{D}_{R/A}$;

iii) $\psi(\partial^{(\alpha)}\overline{r}) = \psi(\partial^{(\alpha)}) \star \psi(\overline{r})$ any $\alpha, r \in R$.

i) We can write, WLOG, $\delta = \overline{s}\partial^{(\alpha)}$.

$$\psi(\overline{rs}\partial^{(\alpha)}) = (-1)^{|\alpha|}\partial^{(\alpha)}\overline{rs} = \left((-1)^{|\alpha|}\partial^{(\alpha)}\overline{s}\right)\overline{r} = \psi(\overline{r}) \star \psi(\overline{s}\partial^{(\alpha)}).$$

ii) Similar.

iii)

$$\begin{aligned}
\psi(\partial^{(\alpha)}\overline{r}) &= \psi\left(\sum_{\beta+\gamma=\alpha} \overline{\partial^{(\beta)}(r)}\partial^{(\gamma)}\right) \\
&= \sum_{\beta+\gamma=\alpha} (-1)^{|\gamma|}\partial^{(\gamma)}\overline{\partial^{(\beta)}(r)} \\
&= \sum_{\beta+\gamma=\alpha} (-1)^{|\gamma|}\left(\sum_{\delta+\epsilon=\gamma} \overline{\partial^{(\delta)}(\partial^{(\beta)}(r))}\partial^{(\epsilon)}\right) \\
&= \sum_{\beta+\gamma+\epsilon=\alpha} (-1)^{|\delta+\epsilon|}\overline{\binom{\beta+\delta}{\beta}\partial^{(\beta+\delta)}(r)}\partial^{(\epsilon)} \\
&= (-1)^{|\alpha|}\sum_{\epsilon+\zeta=\alpha}\left(\sum_{\beta+\delta=\zeta}(-1)^{|\beta|}\binom{\zeta}{\beta}\right)\overline{\partial^{(\zeta)}(r)}\partial^{(\epsilon)}.
\end{aligned}$$

By Lemma 6.1.14, $\sum_{\beta+\delta=\zeta}(-1)^{|\beta|}\binom{\zeta}{\beta} = 0$ for $\zeta \neq 0$ and $\sum_{\beta+\delta=\zeta}(-1)^{|\beta|}\binom{\zeta}{\beta} = 1$ for $\zeta = 0$. So this is just $(-1)^{|\alpha|}\overline{r}\partial^{(\alpha)}$. Thus, ψ is multiplicative, so is a homomorphism. Then,

$$\psi : \mathrm{D}_{R/A}^{op} \longrightarrow \mathrm{D}_{R/A}$$

is also a homomorphism. $\psi^2(\overline{r}) = \overline{r}$, and $\psi^2(\partial^{(\alpha)} = \psi((-1)^{|\alpha|}\partial^{(\alpha)}) = (-1)^{|\alpha|}(-1)^{|\alpha|}\partial^{(\alpha)} = \partial^{(\alpha)}$. Thus, $\psi^2 = id$, on $\mathrm{D}_{R/A}$, so ψ is an isomorphism. □

Conclusion: Flipping multiplicative order and switching sing on $\partial/\partial x_i$ is antiisomorphism of $\mathrm{D}_{R/A}$. This is sometimes called the Fourier transform on $\mathrm{D}_{\mathbb{C}[x]/\mathbb{C}}$.

Now, show some symmetry property for the operators that preserve a principal ideal.

Proposition 6.1.17. *Let A be a ring, $R = A[\underline{x}]$ polynomial ring, and $f \in R$ nonzero-divisor, $D = \mathrm{D}_{R/A}$. Then, the map $\alpha : ((f) :_{\mathrm{D}_{R/A}} (f)) \longrightarrow ((f) :_{\mathrm{D}_{R/A}} (f))^{op}$ given by $\alpha(\delta) = \overline{f}\psi(\delta)\overline{f}^{-1}$ is an isomorphism, where $\psi : \mathrm{D}_{R/A} \longrightarrow \mathrm{D}_{R/A}^{op}$ is the isomorhism given in Proposition 6.1.16.*

Proof. Can check that ψ extends to an isomorhim $\mathrm{D}_{R_f/A} \longrightarrow \mathrm{D}_{R_f/A}^{op}$ by $\psi(\overline{r}\partial^{(\alpha)}) = (-1)^{|\alpha|}\partial^{(\alpha)}\overline{r}$ for $r \in R_f$, any α.

$$\begin{aligned} &\delta \in ((f) :_D (f)) \Leftrightarrow \delta\overline{f}(R) \subseteq fR \\ &\Rightarrow \overline{f}^{-1}\delta\overline{f}(R) \subseteq R \end{aligned}$$

so $\overline{f}^{-1}\delta\overline{f} \in D$. Then,

$$\begin{aligned} D \ni \psi(\overline{f}^{-1}\delta\overline{f}) &= \psi(\overline{f}^{-1}) \star \psi(\delta) \star \psi(\overline{f}) \\ &= \psi(\overline{f})\psi(\delta)\psi(\overline{f}^{-1}) \\ &= \overline{f}\psi(\delta)\overline{f}^{-1}, \end{aligned}$$

So, $\alpha(\delta) \in D$. Then,

$$\alpha(\delta) \cdot (fR) = \overline{f}\psi(\delta)\overline{f}^{-1}(fR) \subseteq \overline{f}\psi(\delta)(R) \subseteq \overline{f}R,$$

so α is well-defined.

Easy to see α is additive.

$$\begin{aligned} \alpha(\delta\epsilon) &= \overline{f}\psi(\delta\epsilon)\overline{f}^{-1} \\ &= \overline{f}\psi(\delta)\psi(\epsilon)\overline{f}^{-1} \\ &= \overline{f}\psi(\delta)\overline{f}^{-1}\overline{f}\psi(\epsilon)\overline{f}^{-1} \\ &= \alpha(\delta) \star \alpha(\epsilon). \end{aligned}$$

So α is a homomorphism. Then,

$$\begin{aligned}\alpha^2(\delta) &= \alpha(\overline{f}\psi(\delta)\overline{f}^{-1}) \\ &= \overline{f}(\overline{f}\psi(\delta)\overline{f}^{-1})\overline{f}^{-1} \\ &= \overline{f}\psi(\overline{f}^{-1})\psi\psi(\delta)\psi(\overline{f})\overline{f}^{-1} \\ &= \overline{f}\overline{f}^{-1}\psi^2(\delta)\overline{f}\overline{f}^{-1} \\ &= \delta.\end{aligned}$$

Thus, α is a isomorphism. □

Remark 6.1.18. Symmetry properties of differential operator rings holds more generally, e.g., for R finitely generated graded K-algebra that is Gorenstein, one has $\mathrm{D}_{R/K} \simeq \mathrm{D}_{R/K}^{op}$. (Quinlan-Gallego).

We conclude:

Theorem 6.1.19 (Tripp). *Let K be a field of characteristic zero, $R = \frac{K[x,y]}{(xy)}$. Then, $((xy) :_{\mathrm{D}_{K[x,y]/K}} (xy))$ is left and right Noetherian, and hence, so is $\mathrm{D}_{R/K}$.*

Let

- K be a field of characteristic zero,
- R be a polynomial ring over K,
- G finite group acting linearly on R with no pseudoreflections.

Theorem 6.1.20 (Wallach). *In this setting, $\mathrm{D}_{R^G/K}$ is D-algebra simple.*

Proof. Let $J \subseteq \mathrm{D}_{R^G}$ be a nonzero two-sided ideal. Let $\delta\mathfrak{n}J\backslash\{0\}$ be of minimal order. Then, for $f \in R^G$, $[\delta, \overline{f}] = \delta\overline{f} - \overline{f}\delta \in J$ and has lower order, so must be zero, thus $\delta = \overline{r} \in J$ for some $r \in R^G$.

We showed that $\mathrm{D}_{R/K}$ is a finitely generated right $\mathrm{D}_{R^G/K}$-module. Using that the same was true for $gr^{\mathrm{ord}}(\mathrm{D}_{R/K})^G \simeq gr^{\mathrm{ord}}(\mathrm{D}_{R^G/K}) \subseteq gr^{\mathrm{ord}}(\mathrm{D}_{R/K})$ by Kantor's theorem.

Write $\mathrm{D}_{R/K} = \sum_i \gamma_i \,\mathrm{D}_{R^G/K}$ for $\gamma_1, \dots, \gamma_t \,\mathrm{D}_{R/K}$ and $N = \max\{\mathrm{ord}(\gamma)\} + 1$.

Set $\gamma_i^{(0)} := \gamma_i$, $\gamma_i^{(j)} := [\gamma_i^{(j-1)}, \overline{r}]$ inductively, so, in particular, $\gamma_i^{(N)} = 0$ for i.

Claim: For each k and any $\delta \in \mathrm{D}_{R/K}$, there are $c_1, \dots, c_k \in \mathbb{Z}$ with $\overline{r}^k\gamma_i = \gamma_i\overline{r}^k + c_1\gamma_i^{(1)}\overline{r}^{k-1} + \cdots + c_k\gamma_i^{(k)}$.

Proof of claim: By induction on k with $k = 0$ trivial.

Note that $\overline{r}\gamma_i^{(j)} = \gamma_i^{(j)}\overline{r} - \gamma_i^{(j+1)}$, so, for inductive step,

$$\begin{aligned}\overline{r}^{k+1}\gamma_i &= \overline{r}\gamma_i\overline{r}^k + c_1\overline{r}\gamma_i^{(1)}\overline{r}^{k-1} + \cdots + c_k\overline{r}\gamma_i^{(k)} \\ &= (\gamma_i\overline{r} - \gamma_i^{(1)})\overline{r}^k + c_1(\gamma_i^{(1)}\overline{r} - \gamma_i^{(2)})\overline{r}^{k-1} + \cdots \\ &= \gamma_i\overline{r}^{k+1} + (c_1 - 1)\gamma_i^{(1)}\overline{r}^k + \cdots + (c_k - c_{k-1})\gamma_i^{(k)}\overline{r} - c_k\overline{r}^{k+1}.\end{aligned}$$

Using observation and claim, we have $\overline{r}^N\gamma_i \in \mathrm{D}_{R/K} \cdot \overline{r}$ (left ideal) for each i.

Now, R is D-algebra simple, i.e., $\mathrm{D}_{R/K}$ is simple. Thus,

$$\begin{aligned} 1 &\in \mathrm{D}_{R/K} \cdot \overline{r}^N \cdot \mathrm{D}_{R/K} \\ &\subseteq \mathrm{D}_{R/K} \cdot \overline{r}^N \left(\sum \gamma_i \, \mathrm{D}_{R^G/K}\right) \\ &\subseteq \mathrm{D}_{R/K} \cdot \overline{r}^N \cdot \mathrm{D}_{R^G/K} . \end{aligned}$$

That is, $1 = \sum_i \alpha_i \overline{r} \beta_i$ $\alpha \in \mathrm{D}_{R/K}$, $\beta \in \mathrm{D}_{R^G/K} = (\mathrm{D}_{R/K})^G$.

Now, consider the map

$$\rho : \mathrm{D}_{R/K} \longrightarrow \mathrm{D}_{R^G/K}$$

given by $\rho(\delta) = \frac{1}{|G|} \sum_{g\in G} g \cdot \delta$. Note that $\rho(1) = 1$.

Further, this is a right $\mathrm{D}_{R^G/K}$-module homomorphism: if $\delta \in \mathrm{D}_{R/K}$, $\epsilon \in \mathrm{D}_{R^G/K}$, then

$$\begin{aligned} \rho(\delta\epsilon) &= \frac{1}{|G|} \sum_{g\in G} g \cdot (\delta\epsilon) \\ &= \frac{1}{|G|} \sum_{g\in G} (g \cdot \delta) \cdot (g\epsilon) \\ &= \frac{1}{|G|} \sum_{g\in G} (g \cdot \delta)\epsilon \\ &= \rho(\delta) \cdot \epsilon. \end{aligned}$$

Thus,

$$\begin{aligned} 1 &= \rho(1) \\ &= \rho\left(\sum_i \alpha_i \overline{r} \beta_i\right) \\ &= \sum_i \rho(\alpha_i \overline{r} \beta_i) \\ &= \sum_i \rho(\alpha_i) \overline{r} \beta_i \\ &\in J. \end{aligned}$$

Thus, $J = \mathrm{D}_{R^G/K}$. □

Corollary 6.1.21. *If R, K, G as above, then any nonzero local cohomology module on R^G is faithful.*

Exercise 6.1.22. *Let $R = \mathbb{C}[x^2, xy, y^2]$. Find explicit operators in $\mathrm{D}_{R/\mathbb{C}}$ that show $1 \in \mathrm{D}_{R/\mathbb{C}} \cdot \overline{x}^2 \cdot \mathrm{D}_{R/\mathbb{C}}$. I.e., find $\alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_t \in \mathrm{D}_{R/\mathbb{C}}$ such that $1 = \sum_{i=1}^t \alpha_i \cdot \overline{x}^2 \cdot \beta_i$.*

6.2 Good filtrations

Definition 6.2.1. Let $(T, F^\bullet)$ be a filtered ring, and M a left (right) T-module. A filtration $G^\bullet$ on M is a good filtration if $gr(M, G^\bullet)$ is a finitely generated $gr(T, F^\bullet)$-module.

Proposition 6.2.2. *Let $(T, F^\bullet)$ be a filtered ring, with $gr(T, F^\bullet)$ finitely generated commutative K-algebra. Then, M is a finitely generated left (right) T-module if and only if M admits a good filtration.*

Proof. ($\Leftarrow$) If M has a good filtration, then $gr(M, G^\bullet)$ finitely generated $gr(T, F^\bullet)$-module. Then, a lift of the generators of $gr(M, G^\bullet)$ to M forms a generating set for M as a T-module.

($\Rightarrow$) Given $\{m_1, \ldots, m_t\}$ generator set for M, set $G^i := \sum_j F^i \cdot m_j$. This is clearly ascending, satisfies $F^a G^b = \sum_j F^a F^b m_j \subseteq \sum_j F^{a+b} m_j = G^{a+b}$, and $\bigcup_i G^i = M$ since $\{m_1, \ldots, m_t\}$ generates.

Show that $gr(M, G^\bullet)$ is finitely generated over $gr(T, F^\bullet)$ (exercise). □

Proposition 6.2.3. *Let $(T, F^\bullet)$ be a filtered K-algebra with $gr(T, F^\bullet)$ finitely generated commutative K-algebra. Let M be a left (right) T-module. Let $G^\bullet$ be a good filtration on M, $H^\bullet$ any filtration on M. Then, there exist $a \in \mathbb{N}$ such that $G^i \subseteq H^{i+a}$ for all i.*

Proof. Pick $m_1, \ldots, m_t \in M$ such that $\overline{m}_1 = m_1 + Ga_1 - 1, \ldots, \overline{m}_t = m_t + Ga_t - 1$ generate $gr(M, G^\bullet)$ as a $gr(T, F^\bullet)$-module.

Let $b_1, \ldots, b_t$ such that $m_i \in H^{b_i} \backslash H^{b_1 - 1}$ for each i. The assumption on generator implies that $G_t = \sum_i F_{t-a_i} \cdot m_i$ for each t. Then, for $t > \max\{a_1\}$,

$$\begin{aligned} G_t &= \sum F_{t-a_i} m_i \\ &\subseteq \sum F_{t-a_i} H^{b_i} \\ &\subseteq \sum H^{t+b_i-a_i} \\ &\subseteq H^{t+a} \end{aligned}$$

for $a = \max\{b_i - a_i\}$ □

Proposition 6.2.4. *Let $(T, F^\bullet)$ be a filtered K-algebra with $gr(T, F^\bullet)$ finitely generated commutative K-algebra, M be a left (right) T-module with $G^\bullet$, $H^\bullet$ good filtrations. Then, exists c such that $G^{i-c} \subseteq H^i \subseteq G^{i+c}$ for all i*

Proof. It follows from last proposition by switching roles and taking maximum. □

Morally, all good filtrations are "some up to a shift." For finitely generated modules, get notation of filtration that is unique/well-defined enough to preserve certain properties/ invariants.

Question 6.2.5. Over $(T, F^\bullet)$ filtered K-algebra with $gr(T, F^\bullet)$ commutative finitely generated K-algebra, M left T-module with good filtration $G^\bullet$. Is any filtration $G^\bullet$ on M good?

Answer: No, e.g., $T = \mathrm{D}_{\mathbb{C}[x]/\mathbb{C}}$ with $F^\bullet$ order filtration. $M = H^1_{(x)}([x])$
$M = \mathrm{D}_{\mathbb{C}[x]/\mathbb{C}} \cdot \left[\frac{1}{x}\right]$ a good filtration is

$$\begin{aligned} G^i &= T^i \cdot \left[\frac{1}{x}\right] \\ &= \mathrm{D}^i_{\mathbb{C}[x]/\mathbb{C}} \cdot \left[\frac{1}{x}\right] \\ &= \bigoplus_{j \leq i} \mathbb{C}[x] \left(\frac{\partial}{\partial x}\right)^j \cdot \left[\frac{1}{x}\right] \\ &= \bigoplus_{a \leq i+1} \mathbb{C}\left[\frac{1}{x^a}\right] \\ &= [M]_{\geq -i-1}. \end{aligned}$$

Filtration condition for $H^\bullet$: $F^i H^j \subseteq H^{i+j}$.

Set

$$\begin{aligned} Hi &:= G^{\lfloor \sqrt{2} i \rfloor} \\ &= \bigoplus_{a \leq \lfloor \sqrt{2} i \rfloor + 1} \mathbb{C}\left[\frac{1}{x^a}\right] \\ &= [M]_{\geq -\lfloor \sqrt{2} i \rfloor - 1}. \end{aligned}$$

$$\begin{aligned} = \mathrm{D}^i_{\mathbb{C}[x]/\mathbb{C}} \cdot H^j &== \mathrm{D}^i_{\mathbb{C}[x]/\mathbb{C}} \cdot G^{\lfloor \sqrt{2} j \rfloor} \\ &\subseteq G^{\lfloor \sqrt{2} j + i \rfloor} \\ &\subseteq G^{\lfloor \sqrt{2} (j+i) \rfloor} \\ &= H^{i+j}. \end{aligned}$$

Exercise 6.2.6. *$gr(M, H^\bullet)$ is not a finitely generated $gr^{\mathrm{ord}}(\mathrm{D}_{\mathbb{C}[x]})$-module.*

6.3 Bernstein filtration

We want to consider smaller filtration on $\mathrm{D}_{K[x]/K}$, K field of characteristic zero.

Definition 6.3.1. For K field of characteristic zero, $R = K[\underline{x}]$, on $\mathrm{D}_{R/K}$, we set

$$B^i = K \cdot \{\delta \in \mathrm{D}_{R/K} \mid \delta \text{ homogeneous and } 2\operatorname{ord}(\delta) + \operatorname{ord}(\delta) \leq i\}.$$

$B^\bullet$ is called the **Bernstein filtration**.

We need to check that this is a filtration: on monomial basis

$$\overline{x}^a \frac{\partial}{\partial x_1}^{b_1} \cdots \frac{\partial}{\partial x_n}^{b_n} = \mu_{a,b}$$

we have

$$\begin{aligned} 2\operatorname{ord}(\mu_{a,b}) + \operatorname{ord}(\mu_{a,b}) &= 2|b| + |a| - |b| \\ &= |a| + |b|. \end{aligned}$$

Thus, $B^i \subseteq B^{i+1}$ and $\bigcup_i B^i = \mathrm{D}_{R/K}$.

If $\alpha \in B^i$, $\beta \in B^j$ homogeneous, then $\deg(\alpha\beta) = \deg(\alpha) + \deg(\beta)$, $\operatorname{ord}(\alpha\beta) \leq \operatorname{ord}(\alpha) + \operatorname{ord}(\beta)$, so $\deg(\alpha\beta) + 2\operatorname{ord}(\alpha\beta) \leq i + j$. Thus, $B^\bullet$ is multiplicative. Concretely,

$$B^i = \bigoplus_{|a|+|b| \leq i} K \cdot \overline{x}^a \frac{\partial}{\partial x_1}^{b_1} \cdots \frac{\partial}{\partial x_n}^{b_n}.$$

Note, that each B^i is a finite dimension K-vector space, whereas for $\mathrm{D}^\bullet$ order filtration, each D^i is a finite rank free R-module.

Compute associated graded:

$$B^i/B^{i-1} \simeq \bigoplus_{|a|+|b| \leq i} K \cdot \left(\overline{x}^a \frac{\partial}{\partial x_1}^{b_1} \cdots \frac{\partial}{\partial x_n}^{b_n} + B^{i-1} \right).$$

Note that

$$\begin{aligned} (\overline{x}_i + B^0)(\frac{\partial}{\partial x_i} + B^0) &= \overline{x}_i \frac{\partial}{\partial x_i} + B^1 \\ &= \left(\frac{\partial}{\partial x_i} \overline{x}_i - \overline{1} \right) + B^1 \\ &= \frac{\partial}{\partial x_i} \overline{x}_i + B^1 \\ &= (\frac{\partial}{\partial x_i} + B^0)(\overline{x}_i + B^0). \end{aligned}$$

Likewise, since $\overline{x}_i$ and $\frac{\partial}{\partial x_i}$, or $\overline{x}_i$ and $\overline{x}_j$, or $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$ for $i \neq j$ commute in $\mathrm{D}_{R/K}$, their images commute in $gr(\mathrm{D}_{R/K}, B^\bullet)$. This gives a map of K-algebras

$$\begin{aligned} K[y_1, \ldots, y_n, z_1, \ldots, z_n] &\longrightarrow gr(\mathrm{D}_{R/K}, B^\bullet) \\ y_i &\longrightarrow x_i + B^0 \\ z_i &\longrightarrow \frac{\partial}{\partial x_i} + B^0. \end{aligned}$$

Have $y_1^{a_1} \cdots y_n^{a_n}, z_1^{b_1} \cdots z_n^{b_n} \longrightarrow \overline{x}^a \frac{\partial}{\partial x_1}^{b_1} \cdots \frac{\partial}{\partial x_n}^{b_n} + B^{|a|+b|-1}$, so the homomorphism is a surjection, is a map of graded Kalgebras. Have same vector space dimension in each graded piece, so this is an isomorphism. Thus, $gr^{Ber}(\mathrm{D}_{R/K}) := gr(\mathrm{D}_{R/K}, B^\bullet)$ is a standard graded polynomial ring in $2n$ variables.

6.4 Dimension and multiplicity for D-modules

Let $R = K[\underline{x}]$ be a polynomial ring, K field of characteristic zero. Let M be a finitely generated D-module. Then, M admits a good filtration compatible with the Berstein filtration, say $G^\bullet$. Note that $G^\bullet$ is not unique. E.g., if $M = \mathrm{D} \cdot \{m_1, \ldots, m_t\}$, $G^\bullet = B^\bullet\{m_1, \ldots, m_t\}$ good. Take some $m' \in G^1$,$G'^\bullet = B^\bullet\{m_1, \ldots, m_t, m'\}$ also good with $m' \in (G')^0$.

$gr(M, G^\bullet)$ is a finitely generated $gr^{Ber}(\mathrm{D}_{R/K})$-module.

Can use theory of Hilbert function: $H(gr(M, G^\bullet), n) = \dim_K([gr(M, G^\bullet)]_{\leq n}$ is a polynomial ring for $n \geq -1$ for some t. The degree of this polynomial id the dimension of $gr(M, G\bullet)$ as a $gr^{Ber}(\mathrm{D}_{R/K})$-module, call it d, and $d!$ times leading coefficient is a positive integer.

Definition 6.4.1. For a filtered module $(M; G^\bullet)$ with each G^i a finite dimension K vector space, we define

$$\dim(M; G^\bullet) := \limsup_{n \to \infty} \frac{\log(\dim_K(G^n))}{\log(n)},$$

and for an integer d,

$$\mathrm{e}_d(M; G^\bullet) := \limsup_{n \to \infty} \frac{d! \dim_K(G^n)}{n^d}.$$

Example 6.4.2. In notation of beginning example, take $H^i = G^{(i^2)}$.

$$\begin{aligned} \dim_K(G^i) = i + 1 &\rightsquigarrow \dim(G^\bullet) = 1 \\ \dim_K(H^i) = i^2 + 1 &\rightsquigarrow \dim(H^\bullet) = 2. \end{aligned}$$

Take

$$J^i = G^{(e^i)} \rightsquigarrow \dim(J^\bullet) = \limsup_{n\to\infty} \frac{\log(e^n+1)}{\log(n)} = \limsup_{n\to\infty} \frac{n}{\log(n)} = \infty.$$

Proposition 6.4.3. *Let $R = K[\underline{x}]$, and K be a field of characteristic zero. Let $M b$ be a finitely generated* D*-module with $G^\bullet$ good filtration w.r.t Berstein filtration. Then,* $\dim(M; G^\bullet) \in \{0, 1, \ldots, 2n\}$*, and for $d = \dim(M, G^\bullet)$,* $\mathrm{e}_d(M, G^\bullet)$ *is a positive integer.*

Proof. Follow from Hilbert function discussion: $\dim_K(G^n) = \sum_{i=0}^n \dim_K(G^i/G^{i-1})$ (where $G^{-1} = 0$) (via SES's

$$0 \longrightarrow G^{i-1} \longrightarrow G^i \longrightarrow G^i/G^{i-1} \longrightarrow 0)$$

so

$$\sum_{i=0}^n \dim_K([gr(M, G^\bullet)]_i) = \dim_K([gr(M, G^\bullet)]_{\le n}) = H(gr(M, G^\bullet), n).$$

So $\dim_K(G^n) = an^d +$ lower order (for $n \ge t$ some t) with $d!a \in \mathbb{N}_{>0}$. Then,

$$\dim(M, G^\bullet) = \limsup_{n\to\infty} \frac{\log(an^d)}{\log(n)} = \limsup_{n\to\infty} \frac{d\log(n) + \log(a)}{\log(n)} = d$$

and $\mathrm{e}_d(M, G^\bullet) = d!a \in \mathbb{N}_{>0}$. □

Theorem 6.4.4. *Let K be a field of characteristic zero, $R = K[\underline{x}]$ be a polynomial ring, M finitely generated* D*-module: If $G^\bullet$, $H^\bullet$ are good filtrations on $M^\bullet$ w.r.t Bernstein filtration, then* $\dim(M, G^\bullet) = \dim(M, H^\bullet)$*, and if we call this value d, then* $\mathrm{e}_d(M, G^\bullet) = \mathrm{e}_d(M, H^\bullet)$*.*

Proof. There exists c such that $G^{n-c} \subseteq H^n \subseteq G^{n+c}$ for all n, so

$$\dim_K(G^{n-c}) \le \dim_K(H^n) \le \dim_K(G^{n+c}).$$

Write $\dim(M, G^\bullet) =: d_G$, $\mathrm{e}_{d_G}(M, G^\bullet) = e_G$ and likewise for H: Since

$$\dim_K(G^n) = \frac{e_G}{d_G!} n^{d_G} + \text{ lower order terms},$$

have

$$\begin{aligned}\dim_K(G^{n+c}) &= \frac{e_G}{d_G!}(n+c)^{d_G} + \text{ lower order terms}\\ &= \frac{e_G}{d_G!}\left(n^{d_G} + \binom{d_G}{1}n^{d_G-1}c + \cdots\right) \text{ lower order terms}\\ &= \frac{e_G}{d_G!}n^{d_G} + \text{ lower order terms,}\end{aligned}$$

same for $-c$.

$$\begin{aligned}\frac{e_G}{d_G!} &= \lim_{n\to\infty}\frac{\dim_K(G^n)}{n^{d_G}}\\ &\le \lim_{n\to\infty}\frac{\dim_K(G^{n-c})}{n^{d_G}}\\ &\le \lim_{n\to\infty}\frac{\dim_K(H^n)}{n^{d_G}}\\ &\le \lim_{n\to\infty}\frac{\dim_K(G^{n+c})}{n^{d_G}}\\ &\le \lim_{n\to\infty}\frac{\dim_K(G^n)}{n^{d_G}}\\ &= \frac{e_G}{d_G!},\end{aligned}$$

so $\lim_{n\to\infty}\frac{\dim_K(H^n)}{n^{d_G}}$ is a (finite) positive integer. Thus, the degree of $\dim_K(H^n)$ (as a polynomial for $n \gg 0$) is d_G, and $\mathrm{e}_{d_H}(M, H^\bullet) = e_G$ □

Definition 6.4.5. Let K be a field of characteristic zero, $R = [\underline{x}]$, and M be a finitely generated D-module. Then, we define $\mathrm{d}(M) := \dim(M, G^\bullet)$, $\mathrm{e}(M) = \mathrm{e}_d(M, G^\bullet)$ for a good filtration $G^\bullet$.

The previous theorem implies this is independents of the choice of $G^\bullet$.

Example 6.4.6. We take $\mathrm{D}_{R/K}$ as a free cyclic module. *Ber* is a good filtration w.r.t *Ber*. $gr^{Ber}(\mathrm{D}_{R/K}) \simeq K[\underline{y}, \underline{z}]$ $2n$ variables std graded. $\mathrm{d}(\mathrm{D}_{R/K}) = 2n$, $\mathrm{e}(\mathrm{D}_{R/K}) = n$.

Example 6.4.7. We take $M = R$. $M \simeq \mathrm{D}\,/\,\mathrm{D}\left\langle \frac{\partial}{\partial x_i}\right\rangle$ cyclic generated by 1. $G^i := B^i \cdot 1$

is a good filtration

$$\begin{aligned} B^i \cdot 1 &= \bigoplus_{|a|+|b|\leq i} \left(K \cdot \overline{x}^a \frac{\partial}{\partial x_1}^{b_1} \cdots \frac{\partial}{\partial x_n}^{b_n} \right)(1) \\ &= \bigoplus_{|a|\leq i} (K \cdot \overline{x}^a)(1) \\ &= \bigoplus_{|a|\leq i} K \cdot x^a \\ &= [R]_{\leq i}. \end{aligned}$$

Then, $\mathrm{d}(R) =$ "usual dimension" of $R = n$, $\mathrm{e}(R) =$ "usual multiplicity" of $R = 1$.

Exercise 6.4.8. *For* $M = \mathrm{D} / \mathrm{D} \left\langle \overline{x}_1 \frac{\partial}{\partial x_2}^{b_2} \cdots \frac{\partial}{\partial x_n}^{b_n} \right\rangle$, *what are* $\mathrm{d}(M), \mathrm{e}(M)$? *Can you recognize* M?

Let K be field of characteristic zero, $R = K[x_1, \ldots, x_n]$ be a polynomial ring, and $D = \mathrm{D}_{R/K}$. For a finitely generated D-module M, define

$$\begin{aligned} \mathrm{d}(M) &:= \text{dimension of } G^\bullet,\ G^\bullet \text{ good filtration on } M \\ &= \dim_{gr(D,B^\bullet)}(gr(M, G^\bullet)), \end{aligned}$$

where $G^\bullet$ is compatible with $B^\bullet$, and $gr(M, G^\bullet)$ is a finitely generated $gr(D, B^\bullet)$-module. Independent of choice of $G^\bullet$.

$$\mathrm{e}(M) := \text{multiplicity of } gr(M, G^\bullet) \text{ as a } gr(D, B^\bullet)(gr(M, G^\bullet),$$

also independent of $G^\bullet$.

Example 6.4.9. Take $M = H^n_{(\underline{x})}(R) \simeq D/D \cdot (\underline{x})$, $M \simeq x_1^{-1} \cdots x_n^{-1} K[x_1^{-1}, \ldots, x_n^{-1}]$ as graded K-vector spaces. This is generated by $\mu = \left[\frac{1}{x_1 \cdots x_n}\right]$. Then, $G^t = B^t \cdot \mu$ is a good filtration.

$$\begin{aligned} \bigoplus_{|a|+|b|\leq t} K \cdot \overline{x}^a \partial^{(b)} \cdot \mu &= \bigoplus_{|a|+|b|\leq t} K \left[\frac{1}{x_1^{1+\beta_1-\alpha_1} \cdots x_n^{1+\beta_n-\alpha_n}} \right] \\ &= [M]_{\geq -n-t}. \end{aligned}$$

Have $\dim_K([M]_{\geq -n-t}) = \dim_K([R]_{\leq t})$, then $\mathrm{d}(H^n_{(\underline{x})}(R)) = \mathrm{d}(R) = n$, and $\mathrm{e}(H^n_{(\underline{x})}(R)) = \mathrm{e}(R) = 1$.

Exercise 6.4.10. *Compute* d, e, *of* R_{x_1} *using the definitions. (will see* $\mathrm{e}(R_x) \neq 1$*).*

Lemma 6.4.11. *In characteristic zero polynomial ring setting, if M is a finitely generated* D*-module with any filtration $F^\bullet$ compatible with Ber., then* $\dim(M, F^\bullet) \geq \mathrm{d}(M)$, *and if* $\dim(M, F^\bullet) = \mathrm{d}(M)$*, then* $\mathrm{e}_{\mathrm{d}(M)}(M, F^\bullet) \geq \mathrm{e}(M)$.

Proof. Boils down any good filtration is contained in a (uniform) shift of an arbitrary filtration. (Exercise). □

Proposition 6.4.12. *In characteristic zero polynomial ring setting, let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of finitely generated D*-modules. Then,* $\mathrm{d}(M) = \max\{\mathrm{d}(L), \mathrm{d}(N)\}$, *and if* $\mathrm{d}(L) = \mathrm{d}(N)$*, then* $\mathrm{e}(M) = \mathrm{e}(L) + \mathrm{e}(N)$.

Proof. Take $G^\bullet$ a good filtration on M w.r.t. Ber.

Claim: $G^\bullet \cap L$ is a good filtration on L, and $G^\bullet N$ is a good filtration on N.

Have seen that there is SES

$$0 \longrightarrow gr(L, G^\bullet \cap L) \longrightarrow gr(M, G^\bullet) \longrightarrow gr(N, G^\bullet N) \longrightarrow 0$$

of $gr(\mathrm{D}_{R/K}, B^\bullet)$. Since $gr(\mathrm{D}_{R/K}, B^\bullet)$ is Noetherian and $gr(M, G^\bullet)$ is finitely generated, so is $gr(L, G^\bullet \cap L)$, so $G^\bullet \cap L$ is a good filtration. $gr(N, G^\bullet N$ is quotient of finitely generated, so it is finitely generated.

Then,

$$\begin{aligned}\mathrm{d}(M) &= \mathrm{d}(gr(M, G^\bullet)) \\ &= \max\{\mathrm{d}(gr(L, G^\bullet \cap L)), \mathrm{d}(gr(N, G^\bullet N))\} \\ &= \max\{\mathrm{d}(L), \mathrm{d}(N)\},\end{aligned}$$

and likewise for $\mathrm{e}(L), \mathrm{e}(M), \mathrm{e}(N)$. □

Lemma 6.4.13. *Let $\delta \in B^t$. Then,* $[\delta, \overline{x}_j], \left[\delta, \frac{\partial}{\partial x_j}\right] \in B^{t-1}$.

Proof. Exercise. □

Theorem 6.4.14 (Bernstein's inequality)**.** *Let K be a field of characteristic zero, $R = [\underline{x}]$ be a polynomial ring of dimension n, and $(M, G^\bullet)$ be a $(\mathrm{D}_{R/K}, B^\bullet)$-module. If $M \neq 0$, then* $\dim(M, G^\bullet) \geq n$*. Thus, if M is finitely generated,* $\mathrm{d}(M) \in \{n, n+1, \ldots, 2n\}$.

Proof. Show by induction over t that the map of vector spaces

$$\begin{aligned}B^t &\longrightarrow \mathrm{Hom}_K(G^t, G^{2t}) \\ \delta &\longrightarrow (m \longrightarrow \delta m)\end{aligned}$$

is injective.

Need to see that $\delta \in B^t \backslash \{0\}$, then $\delta(G^t) \neq 0$. For $t = 0$, $B^0 = K$, so $B^0 \backslash \{0\} = K^\times$.

Inductive step:

If $[\delta, \overline{x}_i] = 0$ for all i, then $\delta = \overline{r} \in \overline{R} \cap B^t$, implies $\left[\delta, \frac{\partial}{\partial x_i}\right] = \overline{\frac{-\partial}{\partial x_i}(r)} \neq 0$ some i unless $\overline{r} \in B^0$(in which case were are done). Can assume that either $[\delta, \overline{x}_i] \neq 0$ or $\left[\delta, \frac{\partial}{\partial x_i}\right] \neq 0$ some i.

If $[\delta, \overline{x}_i] \neq 0$ by inductive hypothesis $(\delta\overline{x}_i - \overline{x}_i\delta)(G^{t-1}) \neq 0$, implies $0 \neq \delta(\overline{x}_i G^{t-1}) \subseteq \delta(G^t)$ or $0 \neq \delta(G^{t-1}) \subseteq \delta(G^t)$.

If $\left[\delta, \frac{\partial}{\partial x_i}\right] \neq 0$ by inductive hypothesis $\left(\delta\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}\delta\right)(G^{t-1}) \neq 0$, implies $0 \neq \delta\left(\frac{\partial}{\partial x_i} G^{t-1}\right) \subseteq \delta(G^t)$ or $0 \neq \delta(G^{t-1}) \subseteq \delta(G^t)$.

This completes the claim.

Thus,

$$\begin{aligned}\binom{2n+t}{t} &= \dim_K(B^t) \\ &\leq \dim_K(\operatorname{Hom}_K(G^t, G^{2t})) \\ &= \dim_K(G^t) \cdot \dim_(G^{2t}),\end{aligned}$$

and $\binom{2n+t}{t} = \frac{t^{2n}}{(2n)!}$+lower order terms. This implies,

$$\begin{aligned}2n &= \limsup_t \frac{\log(t^{2n}/(2n)!)}{\log(t)} \\ &\leq \limsup_t \frac{\log(\dim_K(G^t)\dim_(G^{2t}))}{\log(t)} \\ &= \limsup_t \frac{\log(\dim_K(G^t)) + \log(\dim_(G^{2t}))}{\log(t)}.\end{aligned}$$

But,

$$\begin{aligned}\limsup_t \frac{\log(\dim_(G^{2t}))}{\log(t)} &= \limsup_t \frac{\log(\dim_(G^{2t}))}{\log(t) + \log(2)} \\ &\leq \limsup_t \frac{\log(\dim_(G^{t}))}{\log(t)} \\ &= \mathrm{d}(M, G^\bullet).\end{aligned}$$

Thus, $2\,\mathrm{d}(M, G^\bullet) \geq 2n$. Then, $\mathrm{d}(M, G^\bullet) \geq n$. □

Chapter 7

Holonomicity

Definition 7.0.1. Let K a field of characteristic zero, $R = K[x_1, \dots, x_n]$. A D-module is **holonomic** if it is finitely generated and $\mathrm{d}(M) = n$, or else $M = 0$.

We will say $\mathrm{e}(0) = 0$. If M is holonomic, then $M = 0 \Leftrightarrow \mathrm{e}(0) = 0$. Then, if $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ are holonomic, $\mathrm{e}(M) = \mathrm{e}(L) + \mathrm{e}(N)$.

Example 7.0.2. Some holonomic module are: $0, R, H^n_{(\underline{x})}(R)$ (if $n > 0$). A nonholonomic module is $\mathrm{D}_{R/K}$.

Remark 7.0.3. Submodules, and quotient modules holonomic modules are holonomic: if $N \subseteq M$, $\mathrm{d}(N) \leq \mathrm{d}(M) = n$, so by Bernstein, $N = 0$ of $\mathrm{d}(N) = 0$. N is finitely generated, since M is finitely generated, and $\mathrm{D}_{R/K}$ is left Noetherian. (If T is left Noetherian, then f.g iff Noetherian for left T-modules)

Proposition 7.0.4. *If M is a holonomic* D*-module, then M has finite length as a* D*-module, moreover,* $\ell_{\mathrm{D}_{R/K}}(M) \leq \mathrm{e}(M)$.

Proof. If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ SES of holonomic D-modules, and $\mathrm{e}(L) = \mathrm{e}(M)$, then $\mathrm{e}(N) = 0$, this implies $N = 0$, thus $L = M$. Thus, given a chain of submodules (proper)

$$\cdots \subsetneq M_2 \subsetneq M_1 \subsetneq M$$

(each M_i is necessarily holonomic. We have

$$\mathrm{e}(M) > \mathrm{e}(M_1) > \mathrm{e}(M_2) > \cdots$$

Since these are all positive integer, the chain must have length at most $\mathrm{e}(M)$. ☐

Example 7.0.5. Since R, $H^n_{(\underline{x})}(R)$ are holonomic with $e = 1$, they have length at must 1, so they are simple D-modules.

In general the equality is rare. The equality implies that every composition functor has multiplicity 1.

Question 7.0.6. What can we say about holonomic D-modules of multiplicity 1 $\leftrightsquigarrow$ holonomic D-modules with $\ell = e$?

Let K be a field of characteristic zero $R = K[x_1, \ldots, x_n]$, and M be a finitely generated $\mathrm{D}_{R/K}$-module, then $gr(M, G^\bullet)$ as a $gr^{Ber}(\mathrm{D}_{R/K})$-module for $G^\bullet$ good compatible with Ber.

Consider, $V(\mathrm{Ann}_{gr^{Ber}(\mathrm{D}_{R/K})}(gr(M, G^\bullet))$ the **support variety** of M.

Exercise 7.0.7. *The support variety of a finitely generated* $\mathrm{D}_{R/K}$*-module is independent of the choice of* $G^\bullet$*, but* $\mathrm{Ann}_{gr^{Ber}(\mathrm{D}_{R/K})}(gr(M, G^\bullet)$ *is not.*

$\dim(M) = \dim(V(\mathrm{Ann}_{gr^{Ber}(\mathrm{D}_{R/K})}(gr(M, G^\bullet))))$.

Proposition 7.0.8. *Let* K *be a field of characteristic zero* $R = K[x_1, \ldots, x_n]$*, and* M *be a* $\mathrm{D}_{R/K}$*-module (not a priori finitely generated). If there is a filtration* $F^\bullet$ *on* M *compatible with Ber s.t. there exists* $c > 0 : \dim_K(F^t) \leq ct^n$*, for all* $t \gg 0$*, then* M *is finitely generated, and hence holonomic.*

Proof. Want to show that M is a Noetherian D-module, which implies finitely generated. It suffices to show that every chain of finitely generated submodules stabilizes.

If $0 \neq L \subseteq M$ is finitely generated, then $F^\bullet \cap L$ is a filtration on L of dim $\leq n$ since $\dim_K(F^t \cap L) \leq \dim_K(F^t)$. By Bernstein's inequality, $\dim(L) = n$. Then, $\mathrm{e}_n(F^\bullet \cap L) \leq \mathrm{e}_n(F^\bullet) \leq n!c$. By lemma from last time, $\mathrm{e}(L) \leq \mathrm{e}(F^\bullet \cap L)$, so $\mathrm{e}(L) \leq n!c$.

Thus, if
$$0 \subsetneq L_1 \subsetneq L_2 \subsetneq L_3 \subsetneq \cdots$$
is a chain of finitely generated submodules (proper), we have $\mathrm{e}(L_i) > \mathrm{e}(L_{i-1})$ each i, so $\mathrm{e}(L_i) \leq i$, but $\mathrm{e}(L_i) \leq n!c$, so the chain cannot consist of $n!c$ modules. $\square$

Remark 7.0.9. In the context of last proposition, suffices to show $\dim_K(F^t)$ is bounded by a polynomial of degree n.

Proposition 7.0.10. *Let* K *be a field of characteristic zero* $R = K[x_1, \ldots, x_n]$*. Let* M *be an holonomic module. Then, for any* $f \in R$*,* M_f *is holonomic.*

Proof. Let $G^\bullet$ be a good filtration on M. Let f have degree at most a ($f \in [R_{\leq a}]$). Then, $\overline{f} \in B^a$. Set, $F^t := \frac{1}{f^t} G^{(a+1)t}$. If $\frac{m}{f^t} \in F^t$, so $m \in G^{(a+1)t}$, then $\overline{x}_i \frac{m}{f^t} = \frac{x_i \cdot m}{f^t} = \frac{\overline{\overline{x}}_i f m}{f^{t+1}}$ with $\overline{x}_i f m \in B^{a+1} G^{(a+1)t} \subseteq G^{(a+1)(t+1)}$, this implies $\overline{x}_i \cdot \frac{m}{f^t} \in F^{t+1}$.

Likewise, $\frac{\partial}{\partial x_i} \cdot \frac{m}{f^t} = \frac{f\left(\frac{\partial}{\partial x_i} \cdot m\right) - \left(\frac{\partial}{\partial x_i}(f)\right) \cdot m}{f^{t+a}} \in \frac{1}{f^{t+1}} G^{(a+1)(t+1)} = F^{(t+1)}$. Thus, $B^1 \cdot F^t \subseteq F^{t+1}$. Since $B^s = B^1 \cdots B^1$, get that $F^\bullet$ is compatible with Ber.

Then,

$$\begin{aligned}\dim_K(F^t) &= \dim_K\left(\frac{1}{f^t}G^{(a+1)t}\right)\\ &= \dim_K(G^{(a+1)t})\\ &= \frac{\mathrm{e}(M)}{n!}((a+1)t)^n + \text{lower order terms}\\ &= \frac{\mathrm{e}(M)(a+1)^t}{n!}t^n + \text{L.O.T}\end{aligned}$$

Then by last proposition, M_f is holonomic. □

Theorem 7.0.11. *Let K be a field of characteristic zero $R = K[x_1, \dots, x_n]$, adn M be an holonomic* D*-module. For any ideal $I \subseteq R$, $H^i_I(M)$ is holonomic. In particular, $H^i_I(R)$ is holonomic.*

Proof. The Čech complex on a gen. set for I is a complex of holonomic D-modules. Since submodules and quotient modules of holonomic modules are holonomic, cohomology of Čech complex is holonomic. □

Example 7.0.12. Localization at an arbitrary multiplicative set is not necessarily holonomic. Moreover, is not necessarily finitely generated as D-module. If $R = \mathbb{C}[x]$, $M = \mathbb{C}(X)$, then M is not finitely generated. otherwise write $M = \mathrm{D}_{R/\mathbb{C}} \cdot \left\langle \frac{r_1}{s_1}, \dots, \frac{r_t}{s_t}\right\rangle$. Have $\frac{r_1}{s_1}, \dots, \frac{r_t}{s_t} \subseteq R_{s_1 \cdots s_t}$, which is a $\mathrm{D}_{R/\mathbb{C}}$-module, so $\mathrm{D}_{R/\mathbb{C}} \cdot \left\langle \frac{r_1}{s_1}, \dots, \frac{r_t}{s_t}\right\rangle \subseteq R_{s_1 \cdots s_t}$.

The same argument shows that only D-modules finitely generated localizations are localization of form R_f.

Proposition 7.0.13. *Let $A \longrightarrow R$ rings, R Noetherian, $M \neq 0$ a $\mathrm{D}_{R/K}$-module. If M is simple, then $\mathrm{Ass}_R(M)$ is a singleton.*

Proof. For $P \in \mathrm{Ass}_R(M)$, $H^0_P(M)$ in nonzero, since $R/P \hookrightarrow M$, and

$$\begin{aligned}H^0_P(M) &= \mathrm{Ker}(M \longrightarrow \bigoplus_i M_{f_i})\\ &= \{m \in M \mid \exists t : P^t m = 0\},\end{aligned}$$

so image of R/P in M is contained in $H^0_P(M)$.

Since M is simple and $H^0_P(M)$ is a $\mathrm{D}_{R/A}$-submodule $H^0_P(M) = M$. Then, if $P, Q \in \mathrm{Ass}_R(M)$, there exists $m \in M$ with $Ann_R(m) = Q$, so there exists n such that $P^n \subseteq Ann_R(m) = Q$, so $P \subseteq Q$. Likewise $Q \subseteq P$, so $P = Q$. □

Theorem 7.0.14 (Lyubeznik)**.** *Let K be a field of characteristic zero, $R = K[x_1, \dots, x_n]$. Then, any holonomic* D*-module M has finitely many associated primes as an R-module. In particular, any $H^i_I(R)$ has finitely many associated primes.*

Proof. Take a filtration of M by simple D-modules (can do since $\ell_{\mathrm{D}_{R/K}}(M) < \infty$)

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\ell = M.$$

Have $\mathrm{Ass}_R(M_i) \subseteq \mathrm{Ass}_R(M_{i-1}) \cup \mathrm{Ass}_R(M_i/M_{i-1})$.

By induction on i, each M_i, in particular $M_\ell = M$, has finitely many associated primes. □

This is not true in general for Noetherian rings!

Example 7.0.15 (Singh)**.** Let $R = \frac{\mathbb{Z}[u,v,w,x,y,z]}{(ux+vy+wz)}$ will show that $H^3_{(u,v,w)}(R)$ has infinitely many associated primes. We are looking at

$$R_{uv} \oplus R_{vw} \oplus R_{uw} \longrightarrow R_{uvw} \longrightarrow 0$$

so

$$H_{(u,v,w)}(R) = \frac{R_{uvw}}{in(R_{uv} \oplus R_{uw} \oplus R_{vw})}.$$

Can write any element as $\left[\frac{r}{(uvw)^t}\right]$ some $r \in R$ $t \in \mathbb{N}$.

We have

$$\begin{aligned}\left[\frac{r}{(uvw)^t}\right] = 0 &\Leftrightarrow \frac{r}{(uvw)^t} = \frac{r_1}{(uv)^a} + \frac{r_2}{(uw)^b} + \frac{r}{(vw)^c} \ \exists r_1, r_2, r_3 \in R,\ \exists a,b,c \in \mathbb{N} \\ &\Leftrightarrow \frac{r}{(uvw)^t} = \frac{r_1}{(uv)^{t+k}} + \frac{r_2}{(uw)^{t+k}} + \frac{r}{(vw)^{t+k}} \ \exists r_1, r_2, r_3 \in R,\ \exists k \in \mathbb{N} \\ &\Leftrightarrow r(uvw)^k = r_1 w^{t+k} + r_2 v^{t+k} + r_3 u^{t+k} \ \exists r_1, r_2, r_3 \in R,\ \exists k \in \mathbb{N} \\ &\Leftrightarrow \exists k \in \mathbb{N} : r(uvw)^k \in (u^{t+k}, v^{t+k}, w^{t+k}).\end{aligned}$$

Last time: For K a field of characteristic zero, $R = K[\underline{x}]$ a polynomial ring, $H^i_I(R)$ has finite length as $\mathrm{D}_{R/K}$-module, then $\mathrm{Ass}_R(H^i_I(R))$ is finite.

Example 7.0.16. Let $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$. Then, $H^1_{(x)}(R) = Rx/R$ does not have finite length as a $\mathrm{D}_{R/\mathbb{C}}$-module: $H^1_{(x)}(R)$ is graded, and has elements of arbitrarily low negative degree: $\left[\frac{1}{x^t}\right]$.

Note that $\delta \in [\mathrm{D}_{R(\mathbb{C}}]_i$ acts on R_x as a map of degree i: $\delta(r/x^t) = \sum_{j=0}^{\mathrm{ord}(\delta)} \frac{\delta^{(j)}(r)}{(x^t)^{j+1}}$, where $\delta^{(j)} = [\delta^{(j-1)}, \overline{x}^t]$. Then, $\delta^{(j)}$ has degree $i + jt$, so

$$\begin{aligned}|\delta(r/x^t)| &= |r| + i + jt - (tj + t) \\ &= |r| + i - t \\ &= |r/x^t| + i.\end{aligned}$$

Likewise, $\delta \in [\mathrm{D}_{R/\mathbb{C}}]_i$ acts on $H^1_{(x)}(R)$ as a map of degree i. But, $[\mathrm{D}_{R/\mathbb{C}}]_{<0} = 0$. Thus, $H^1_{(x)}(R)$ is not finitely generated as a D-module. (Otherwise, look at lowest degree of an element in a generating set, any thing generated by that set lives in larger degrees).

Recall: If $I = (f_1, \dots, f_t) \subseteq R$ (Noetherian commutative ring), then any element in $H^t_I(R)$ can be written as an equivalent class $\left[\frac{r}{(f_1\cdots f_t)^k}\right]$, and

$$\left[\frac{r}{(f_1\cdots f_t)^k}\right] = 0 \Leftrightarrow \exists \ell : r(f_1\cdots f_t)^\ell \in (f_1^{k+\ell}, \dots, f_t^{k+\ell})$$

Example 7.0.17 (Singh)**.** $R = \frac{\mathbb{Z}[u,v,w,x,y,z]}{(ux+vy+wz)}$ want to show that $H^3_{(u,v,w)}(R)$ has infinitely many associated primes.

Key claim:(for each prime p) Let

$$\begin{aligned}\lambda_p &= \frac{(ux)^p + (vy)^p + (wz)^p - (ux+vy+wz)^p}{p} \\ &= -\sum_{i+j+k=p; i,j,k\neq p} \frac{\binom{p}{i,j,k}}{p}(ux)^i(vy)^j(wz)^k \in R.\end{aligned}$$

The element $\mu_p = \left[\frac{\lambda_p}{u^pv^pw^p}\right]$ is nonzero in $H^3_{(u,v,w)}(R)$, but $p\mu_p = 0$ in $H^3_{(u,v,w)}(R)$. $p\mu_p = 0$ since $p\lambda_p = (ux)^p + (vy)^p + (wz)^p \in (u^p, v^p, w^p)$. Need to check that $\lambda_p(uvw)^\ell \notin (u^{p+\ell}, v^{p+\ell}, w^{p+\ell})R$ for all $\ell \geq 0$.

Suppose otherwise. Can give R a $\mathbb{Z}^3$-grading

$$\begin{gathered}|u| = (1,0,0) \quad |v| = (0,1,0) \quad |w| = (0,0,1) \\ |x| = (-1,0,0) \quad |y| = (0,-1,0) \quad |z| = (0,0,-1)\end{gathered}$$

(ok, since $ux + vy + wz$ is homogeneous). Then, $|\lambda_p| = 0$, $|\lambda_p(u,v,w)^\ell| = (\ell,\ell,\ell)$. $\lambda_p(uvw)^\ell = Au^{p+\ell} + Bv^{p+\ell} + Cw^{p+\ell}$ WLGO A, B, C homogeneous $\Rightarrow$

$$\begin{aligned}|A| = (-p,\ell,\ell) &\Rightarrow A = x^pv^\ell w^\ell \cdot A' \\ |B| = (\ell,-p,ell) &\Rightarrow B = u^\ell y^p w^\ell \cdot B' \\ |C| = (\ell,\ell,-p) &\Rightarrow C = u^\ell v^\ell z^p \cdot C'\end{aligned}$$

with A', B', C' of degree $\underline{0}$.

$$\lambda_p(uvw)^\ell = A'(uvw)^\ell(ux)^p + B'(uvw)^\ell(vy)^p + C'(uvw)^\ell(wz)^p,$$

then

$$\lambda_p \in ((ux)^p, (vy)^p, (wz)^p)R_{\underline{0}}$$

where $R_{\underline{0}}$ = subring of R of degree $\underline{0}$ elts $= \frac{\mathbb{Z}[ux,vy,wz]}{(ux)+(vy)+(wz)} \simeq \mathbb{Z}[ux, vy]$.

In $R_{\underline{0}} \simeq \mathbb{Z}[ux, vy]$,

$$\lambda_p = -\sum_{i+j+k=p; i,j,k\neq p} \frac{\binom{p}{i,j,k}}{p}(ux)^i(vy)^j(-ux-vy)^k$$

the coefficient of $(ux)^{p-1}(vy)$ is nonzero mod p, so

$$\begin{aligned}\lambda_p \notin &((ux)^p, (vy)^p, (-ux-vy)^p, p)\mathbb{Z}[ux, vy]\\ &= ((ux)^p, (vy)^p, p)\mathbb{Z}[ux, vy].\end{aligned}$$

Thus, $\lambda_p \notin ((ux)^p, (vy)^p, (wz)^p)\frac{\mathbb{Z}[ux,vy,wz]}{(ux+vy+wz)}$

Now, $R \cdot \mu_p$ is a nonzero submodule with annihilator $\ni p$, so it has an associated prime containing p.

Thus, $\exists Q_p \in \mathrm{Ass}_R(H^3_{(u,v,w)}(R))$ with $p \in Q_p$, for each prime p. These must be distinct (since $p, p' \in Q \Rightarrow 1 \in Q$).

7.1 The D-module $R_f[s] \cdot \underline{f}^s$

Theorem 7.1.1 (Bernstein)**.** *For $f \neq 0$ in $K[x_1, \dots, x_n]$, K a field of characteristic zero, there is a nonzero functional equation for f.*

Proof. Consider $R_f(s) \cdot \underline{f}^s$ as $\mathrm{D}_{R(s)/K(s)}$-module. Consider the descending chain of submodules

$$\mathrm{D}_{R(s)/K(s)} \cdot \underline{f}^s \supseteq \mathrm{D}_{R(s)/K(s)} \cdot f\underline{f}^s \supseteq \mathrm{D}_{R(s)/K(s)} \cdot f^2\underline{f}^s \supseteq \cdots$$

Since $\mathrm{D}_{R(s)/K(s)} \cdot \underline{f}^s$ is holonomic, it has finite length, so it is artinian, so the chain stabilizes. Thus, for some $t \in \mathbb{N}$, $f^t\underline{f}^s \in \mathrm{D}_{R(s)/K(s)} \cdot f^{t+1}\underline{f}^s$. That is, there exist $\widetilde{P}'(s) \in \mathrm{D}_{R(s)/K(s)}$ such that $\widetilde{P}'(s)f^{t+1}\underline{f}^s = f^t\underline{f}^s$ in $R_f(s) \cdot \underline{f}^s$. Write $\widetilde{P}'(s) = \frac{P'(s)}{b'(s)}$ with $\widetilde{P}'(s) \in \mathrm{D}_{R/K}[s]\backslash\{0\}, b'(s) \in K[s]\backslash\{0\}$, so $P'(s)f^{t+1}\underline{f}^s = b'(s)f^t\underline{f}^s$ in $R_f(s) \cdot \underline{f}^s$.

Set $P(s) = P'(s-t), b(s) = b(s-t)$. Then,

$$\begin{aligned}\pi_k(P(s)f\underline{f}^s - b(s)\underline{f}^s) &= P'(k-t)f^{k+1} - b'(k-t)f^k \; k \in \mathbb{Z}\\ &= \pi_{k-t}((P'(s)f^{t+1}\underline{f}^s - b'(s)f^t\underline{f}^s)\\ &= 0\end{aligned}$$

Since this holds for all $k \in \mathbb{Z}$, $P(s)f\underline{f}^s = b(s)\underline{f}^s$ in $R_f(s) \cdot \underline{f}^s$. □

Suppose that for $f \in R$ we have functional equations

$$\begin{aligned}P_1(s)f\underline{f}^s &= b_1(s)\underline{f}^s\\ P_2(s)f\underline{f}^s &= b_2(s)\underline{f}^s\end{aligned}$$

and $c(s) \in K[s]$. Then, $(P_1(s) + P_2(s))f\underline{f}^s = (b_1(s) + b_2(s))\underline{f}^s$ and $c(s)P_1(s)f\underline{f}^s = c(s)b_1(s)\underline{f}^s$. Thus, $\{b(s) \mid \exists$ functional equation for f with $b(s)$ and some $P(s)\}$ is an ideal in $K[s]$, and is nonzero by the theorem.

Definition 7.1.2. The **Bernstein-Sato** polynomial of $f \in K[\underline{x}]$ (K field of characteristic zero) is the monic generator of the ideal $\{b(s) \mid \exists P(s) \in \mathrm{D}_{R/K}[s] \text{ with } P(s)f\underline{f}^s = b(s)\underline{f}^s\}$. Denote $b_f(s)$.

Proposition 7.1.3. *If $f \in R$ is not a unit, then $(s+1)|b_f(s)$.*

Proof. Equivalent to show that for any functional equation for f, $s = -1$ is a root. Indeed, $P(-1) \cdot f^0 = b(-1) \cdot f^{-1}$, but $P(-1) \cdot f^0 \in R$, while $f^{-1} \notin R$, so $b(-1) = 0$. □

Example 7.1.4. 1) For $x_i \in K[\underline{x}]$, $b_{x_i}(s) = s+1$, coming from $\frac{\partial}{\partial x_i} \cdot x_i \underline{x_i}^s = (s+1)\underline{x_i}^s$.

2) For $x_i^n \in K[\underline{x}]$, $b_{x_i^n}(s) = (s+1)(s+\frac{n-1}{n})\cdots(s+\frac{1}{n})$. Coming from $\frac{1}{n^n}\left(\frac{\partial}{\partial x_i}\right)^n \cdot x_i^n \underline{x_i}^{ns} = b_{x_i^n}(s)\underline{x_i}^{ns}$.

3) For $f = x_1^2 + x_2^3$, $b_f(s) = (s+1)(s+5/6)(s+7/6)$.

Exercise 7.1.5. *For x_1x_2+1, find a functional equation and find Bernstein-Sato polynomial.*

Theorem 7.1.6 (Kashiwara)**.** *For K a field of characteristic zero, $R = K[\underline{x}]$, $b_f(s) = \prod_i (s + p_i/q_i)$ $p_i, q_i \in \mathbb{N}_{>0}$ so each is strictly negative and rational.*

In particular, zero is not root, not is any positive integer, so this strengthens the fact that R is D-module simple: $f \in R$

$$P(s) \cdot f\underline{f}^s = b_f(s)\underline{f}^s$$
$$P(0) \cdot f = b_f(0) \in \mathbb{Q}\backslash\{0\}.$$

($\Rightarrow R$ is D-module simple)

For $t \in \mathbb{N}$, have

$$P(0)\cdots P(t-2)P(t-1)f^t = b(t-1)b(t-2)\cdots b(0) \in \mathbb{Q}\backslash\{0\}.$$

Remark 7.1.7. Can also characterize $b_f(s)$ as monic polynomial of minimal degree such that there exist $P(s) \in \mathrm{D}_{R/K}[s]$ with $P(t) \cdot f^{t+1} = b(t) \cdot f^t$ for all $t \in \mathbb{Z}$ (in R_f).

Can ask if we can find solutions to functional equation in this sense (in R_f, for all $t \in \mathbb{Z}$) over any ring R.