Th'm [Bernstein]: For $f \neq 0$ in $K[x_1, \dots, x_n]$, K field of char 0, there is a nonzero functional equation for f.

pf: Consider $R_f(s) \cdot \underline{f}^s$ as $D_{R(s)|K(s)}$-module.
Consider the descending chain of submodules:
$$D_{R(s)|K(s)} \cdot \underline{f}^s \supseteq D_{R(s)|K(s)} \cdot f \underline{f}^s \supseteq D_{R(s)|K(s)} \cdot f^2 \underline{f}^s \supseteq \cdots$$
Since $D_{R(s)|K(s)} \cdot \underline{f}^s$ is holonomic, it has finite length, so it is artinian, so the chain stabilizes. Thus, for some $t \in \mathbb{N}$,
$$f^t \underline{f}^s \in D_{R(s)|K(s)} \cdot f^{t+1} \underline{f}^s.$$
That is, $\exists\ \tilde{P}'(s) \in D_{R(s)|K(s)}$ s.t.
$$\tilde{P}'(s) \cdot f^{t+1} \underline{f}^s = f^t \underline{f}^s \quad \text{in } R_f(s) \cdot \underline{f}^s.$$
Write $\tilde{P}'(s) = \dfrac{P'(s)}{b'(s)}$ $\quad \boxed{D_{R(s)|K(s)} = (K[s] \setminus \{0\})^{-1} D_{R|K}[s]}$

with $P'(s) \in D_{R|K}[s] \setminus \{0\}$, $b'(s) \in K[s] \setminus \{0\}$

So, $P'(s) \cdot f^{t+1} \underline{f}^s = b'(s) \cdot f^t \underline{f}^s$

$\rotatebox{90}{$\in$}$ $D_{R|k}[s]$. in $R_f[s] \cdot \underline{f}^s$.

Set $P(s) = P'(s-t)$, $b(s) = b'(s-t)$.

Then, $\pi_k\left(P(s) f \underline{f}^s - b(s) \underline{f}^s\right)$ $\quad k \in \mathbb{Z}$

$$= P'(k-t) f^{k+1} - b'(k-t) f^k$$

$$= \pi_{k-t}\left(P'(s) f^{t+1} \underline{f}^s - b'(s) f^t \underline{f}^s\right)$$

$$= 0.$$

Since this holds for all $k \in \mathbb{Z}$,

$P(s) f \underline{f}^s = b(s) \underline{f}^s$ in $R_f[s] \cdot \underline{f}^s$. $\boxtimes$

Suppose that for $f \in R$ we have functional equations

$$P_1(s)\cdot f\underline{f}^s = b_1(s)\cdot \underline{f}^s \qquad \text{and } c(s)\in K[s]$$
$$P_2(s)\cdot f\underline{f}^s = b_2(s)\,\underline{f}^s.$$

Then $(P_1(s)+P_2(s))\, f\underline{f}^s = (b_1(s)+b_2(s))\,\underline{f}^s$

and $c(s)\, P_1(s) f\underline{f}^s = c(s)\, b_1(s)\, \underline{f}^s$.

Thus, $\{ b(s) \mid \exists$ functional equation for f with $b(s)$ and some $P(s) \}$

is an ideal in $K[s]$, and is nonzero by the theorem.

for $f\in K[\underline{x}]$, K field of char 0

Def: The Bernstein-Sato polynomial of $f\in K[\underline{x}]$ (K field of char 0) is the monic generator of the ideal

$$\{ b(s) \mid \exists\, P(s)\in D_{R|K}[s] \text{ with } P(s)\cdot f\underline{f}^s = b(s)\underline{f}^s \}.$$

Denote $b_f(s)$.

Prop: ~~If~~ $f \in R$ is not a unit, then $(s+1) \mid b_f(s)$.

Pf: Equivalent to show that for any functional equation for f, $s = -1$ is a root. Indeed,

$P(-1) \cdot f^0 = b(-1) \cdot f^{-1}$, but $P(-1) \cdot f^0 \in R$, while $f^{-1} \notin R$, so $b(-1) = 0$. ∎

Ex: 1) For $x_i \in K[\underline{x}]$, $b_{x_i}(s) = s+1$ coming from $\frac{\partial}{\partial x_i} \cdot x_i \underline{x_i^s} = (s+1) \cdot \underline{x_i^s}$.

2) For $x_i^n \in K[\underline{x}]$, $b_{x_i^n}(s) = (s+1)(s+\frac{n-1}{n}) \cdots (s+\frac{1}{n})$.

coming from $\frac{1}{n^n}\left(\frac{\partial}{\partial x_i}\right)^n \cdot x_i^n \underline{x_i^{ns}} = b_{x_i^n}(s) \cdot \underline{x_i^{ns}}$.

3) For $f = x_1^2 + x_2^3$, $b_f(s) = (s+1)(s+5/6)(s+7/6)$.

<u>Exercise</u>: For x_1x_2+1, find a functional equation & find Bernstein-Sato polynomial.

<u>Theorem</u> [Kashiwara]: For k field of char 0, $R = k[\underline{x}]$, $b_f(s) = \prod_i (s + p_i/q_i)$

$p_i, q_i \in \mathbb{N}_{>0}$ so each ~~root~~ is strictly negative and rational.

In particular, zero is not a root, nor is any positive integer, so this strengthens the fact that R is D-module simple.

$$f \in R \rightsquigarrow P(s)\cdot f\underline{f}^s = b_f(s)\underline{f}^s$$

$$P(0)\cdot f = b_f(0) \in \mathbb{Q}\setminus\{0\}.$$

($\Rightarrow R$ is D-module simple)

For $t \in \mathbb{N}$, have

$$P(0) \cdots P(t-2)P(t-1)\, f^t = \underbrace{b(t-1)b(t-2)\cdots b(0)}_{\in \mathbb{Q}\setminus\{0\}}.$$

<u>Note</u>: Can also characterize $b_f(s)$ as monic poly. of minimal degree such that $\exists\, P(s) \in D_{R|K}[s]$ with

$$P(t)\cdot f^{t+1} = b(t)\cdot f^t \text{ for all } t \in \mathbb{Z}$$

(in R_f).

Can ask if we can find solutions to functional equation in this sense (in R_f, for all $t \in \mathbb{Z}$) over any ring R.

Ex: For $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, if $f \in R$

has positive degree, there are no nonzero solutions to the functional equation.

if we have $P(t) \cdot \underbrace{f^{t+1}}_{(t+1)\cdot|f|} = \underbrace{b(t)}_{deg=0} \cdot \underbrace{f^t}_{deg=t\cdot|f|}$

with $P(s) \in D_{R|\mathbb{C}}[s]$, $b(s) \in \mathbb{C}[s]$,

must have $|P(t)| = -|f|$ for all t,

but $[D_{R|\mathbb{C}}]_{<0} = 0$.

Only have $P(t) = 0$, $b(t) = 0$ as solution.

Ex: For $R = \frac{\mathbb{C}[x,y]}{(xy)}$, $f = x$.

Have $x\left(\frac{\partial}{\partial x}\right)^2 \in D_{R|\mathbb{C}}$.

$$x\left(\frac{\partial}{\partial x}\right)^2 \cdot x^{t+1} = t(t+1)\,x^t$$

So, $P(s) = x\left(\frac{\partial}{\partial x}\right)^2$, $b(s) = s(s+1)$
yield a ~~nonzero~~ functional equation
for f.
Note that $s=-1$ has to be a root,
and so does $s=0$, because
$x \in (x)$ a D-ideal of R,
while $1 \notin (x)$,
so $P(0)\cdot x = b(0) \Rightarrow b(0) = 0$.
See that $b(s) = s(s+1)$ is minimal monic
polynomial appearing in a functional equation.

Differential direct summands

Let $R \hookrightarrow S$ be A-algebras, R direct summands of S with splitting β.

Have seen that $\delta \in D^n_{S|A} \Rightarrow \beta \circ \delta|_R \in D^n_{R|A}$.

Def [ÀM-H-NB]: Let R, S, β be as above. Let M be a $D_{R|A}$-module, N a $D_{S|A}$-module. Suppose further that $M \subseteq N$ with abelian group splitting θ. Say M is a <u>DDS</u> of N via θ if for all $\delta \in D_{S|A}$ and $m \in M$

$$(\beta \circ \delta|_R) \cdot m = \theta(\delta \cdot m)$$

$D_{R|A}$-action on M (for $\cdot$ on the left)

$D_{S|A}$-action on N $(m \in M \subseteq N)$ (for $\cdot$ on the right)

Might write (M, N, θ) is a DDS.

$(\beta \circ \delta|_R) \underset{\uparrow}{\cdot} m = \theta(\delta \underset{\uparrow}{\cdot} m)$

$D_{R|A}$-action on M $\qquad$ $D_{S|A}$-action on N $(m \in M \subseteq N)$

<u>Ex</u>: (R, S, β) is a DDS.

$$(\beta \circ \delta|_R) \underset{\uparrow}{\cdot} r = \beta(\delta \underset{\uparrow}{\cdot} r)$$

$D_{R|A}$-action on R $\qquad$ $D_{S|A}$-action on S

$(M_1, N_1, \theta_1), (M_2, N_2, \theta_2)$ DDS

<u>Def</u>: An S-module homomorphism $\varphi: N_1 \to N_2$ yields a <u>DDS morphism</u> if $\varphi(M_1) \subseteq M_2$, and the diagram

$$\begin{array}{ccc} N_1 & \xrightarrow{\varphi} & N_2 \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

commutes.

<u>Prop</u>: If (M, N, θ) is a DDS, and $f \in R$, then $\theta_f = \theta \otimes_R R_f \rightsquigarrow (M_f, N_f, \theta_f)$ is a DDS, and

$(M, N, \theta) \to (M_f, N_f, \theta_f)$ DDS morphism.

$$(\beta\circ\delta|_R)\underset{\substack{\uparrow\\ D_{R|A}\text{-action}\\ \text{on } M}}{\cdot} m = \theta(\delta \underset{\substack{\uparrow\\ D_{S|A}\text{-action}\\ \text{on } N\ (m\in M\subseteq N)}}{\cdot} m)$$

Pf: Clear the diagram commutes; just check that it's a DDS.

For simplicity, take $M=R$, $N=S$.

$$(\beta\circ\delta|_R)(r/f^t) = \sum_{j=0}^{\mathrm{ord}(\delta)} \frac{(\beta\circ\delta|_R)^{(j)}\cdot(r)}{(f^t)^{j+1}}$$

where $(-)^{(j)} = [(-)^{(j-1)}, f^t]$.

$$\theta(\delta(r/f^t)) = \theta\left(\sum_{j=0}^{\mathrm{ord}(\delta)} \frac{\delta^{(j)}(r)}{(f^t)^{j+1}}\right)$$
$$= \sum_{j=0}^{\mathrm{ord}(\delta)} \frac{\beta(\delta^{(j)}(r))}{f^{t(j+1)}}$$

suffices to show: $(\beta\circ\delta|_R)^{(j)} \overset{?}{=} (\beta\circ\delta^{(j)})|_R$.

by induction on j, $j=0$ trivial.

$$[(\beta\circ\delta|_R)^{(j-1)}, f^t] \overset{IH}{=} [(\beta\circ\delta^{(j-1)})|_R, f^t]$$
$$= \beta\circ\delta^{(j-1)}|_R\circ f^t - f^t\circ\beta\circ\delta^{(j-1)}|_R$$
$$= \beta\circ[\delta^{(j-1)}|_R, f^t] = \beta\circ\delta^{(j)}|_R. \quad \blacksquare$$

Thm [Àlvarez Montaner – Huneke – Núñez Betancourt]:
Let $R \subseteq S$ poly/field of char 0.
Then $\exists P(s) \in D_{R|K}[s]$ and $b(s) \in K[s]$ nonzero s.t. $P(t) \cdot f^{t+1} = b(t) \cdot f^t$ for all $t \in \mathbb{Z}$ (in R_f).

Pf: Have functional equation in S:

$$\tilde{P}(t) \cdot f^{t+1} = \tilde{b}(t) f^t \text{ for all } t \in \mathbb{Z} \text{ in } S_f$$

$\tilde{P}(s) \in D_{S|K}[s]$, $\tilde{b}(s) \in K[s]$.

Apply $\Theta_f = \beta \otimes_R R_f$:

$$\Theta_f(\tilde{P}(t) \cdot f^{t+1}) = \Theta_f(\tilde{b}(t) f^t) \quad \text{for all } t \in \mathbb{Z} \text{ in } R_f$$

$$(\beta \circ \tilde{P}(t)|_R) \cdot f^{t+1} \qquad \tilde{b}(t) f^t$$

Taking $P(s) = \beta \circ \tilde{P}(s)|_R$ and $b(s) = \tilde{b}(s)$, we get a nonzero functional equation.

Cor: There is a nonzero functional equation for $f \in R$ with polynomial $b_{f \in S}(s)$ (BS poly for f considered as an element in S).

Can use same ideas to show for R direct summand of poly ring of char 0, for any ideal $I \subseteq R$, $H^i_I(R)$ has finite length as a $D_{R|K}$-module, and hence has finitely many associated primes.

$$D_{R|K}[S] \longrightarrow D_{R|K}[\partial_t t] \subseteq D_{R[t]|K}$$

$$\circlearrowleft \qquad\qquad\qquad\qquad \circlearrowleft$$

$$R_f[S] \cdot f^S \longrightarrow H^1_{(f-t)}(R[t]_f).$$

Reinterpret functional equation in terms of

$D_{R|K}[-\partial^{(1)}t, -\partial^{(2)}t^2, -\partial^{(3)}t^3, \ldots]$

$\longrightarrow$ Mustață, T. Bitoun, Quinlan-Gallego