$R = \dfrac{\mathbb{Z}[u,v,w,x,y,z]}{(ux+vy+wz)}$ found p-torsion in local cohomology for all p.

If $\mathbb{Z} \rightsquigarrow K$ field, then every LC module has finitely many associated primes.

idea: R_t where $t \in \{u, \dots, z\}$ is $\simeq$ localization of poly ring, so if H is a LC module of R, H_t has fin many assoc primes $\Rightarrow$ OK.

But, there are examples of K-algebras with LC modules with ∞ assoc. primes, e.g.,

$$H^2_{(u,v)}\left(\frac{K[s,t,u,v,w,x]}{s(ux)^2+t(ux)(vy)+s(vy)^2}\right)$$ has ∞ assoc. primes.

Current topic of interest: ~~for~~ which rings R do we have all $H^i_I(R)$ have finitely many assoc. primes?

Conj [Huneke, Lyubeznik]: If R is regular, then all $H^i_I(R)$ have finitely many assoc. primes.

Known for: • poly rings over fields of char 0 [Lyubeznik]
• reg rings of char $p>0$ [Huneke-Sharp]
• smooth $\mathbb{Z}$-algebras [Bhatt-Blickle-Lyubeznik-Singh-Zhang].

Last time: $R_f[s]\cdot \underline{f}^s \longrightarrow R_f$

$s \longmapsto t \in \mathbb{Z}$

(for R poly ring over k field of char 0).

Have: $R_f[s]\cdot \underline{f}^s$ free cyclic $R_f[s]$-module

$$\frac{\partial}{\partial x_i}\cdot \underline{f}^s = \frac{s\frac{\partial f}{\partial x_i}}{f}\cdot \underline{f}^s$$

These determine at most one structure of $D_{R_f|K}[S]$-module,
since $D_{R_f|K}[S] = R_f[S]\cdot\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\rangle$
and $\underline{f}^S$ generates $R_f[S]\cdot \underline{f}^S$
as $D_{R_f|K}[S]$-module.

<u>Alternative construction</u>:

K field of char 0, R K-alg.

Let $D_{R|K}[S] \xrightarrow{\varphi} D_{R[t]|K}$ (S, t indeterminates)

be the map that is identity on $D_{R|K}$

and $S \longmapsto -\frac{\partial}{\partial t}\cdot t$

can identify the image of φ with

$D_{R|K}[-\frac{\partial}{\partial t}\cdot t] \subseteq D_{R[t]|K}$, and φ is injective.

Consider the map

free cyclic $R_f[S]$-module $\qquad\qquad \dfrac{(R[t]_f)_{f-t}}{R[t]_f}$

$$\underbrace{R_f[S]\cdot \underline{f}^S}_{\text{free cyclic } R_f[S]\text{-module}} \xrightarrow{\ \Psi\ } H^1_{(f-t)}(R[t]_f).$$

$$g(S)\cdot \underline{f}^S \longmapsto g\left(-\frac{\partial}{\partial t}\cdot t\right)\left[\frac{1}{f-t}\right].$$

Exercise: Ψ is injective,

so Ψ induces an isomorphism onto the image $D_{R_f|K}\left[\frac{\partial}{\partial t}\cdot t\right]\cdot\left[\frac{1}{f-t}\right]$.

Define $D_{R_f|K}[S]$ structure on $R_f[S]\cdot \underline{f}^S$ as follows:

$$P(S)\cdot g(S)\underline{f}^S := \Psi^{-1}\left(\varphi(P(S))\cdot\Psi(g(S)\underline{f}^S)\right)$$

for $P(S)\in D_{R_f|K}[S]$ and $g(S)\in R_f[S]$.

For $h(S)\in R_f[S]$,

$$\begin{aligned} h(S)\cdot g(S)\underline{f}^S &= \Psi^{-1}\left(\varphi(h(S))\cdot\Psi(g(S)\underline{f}^S)\right) \\ &= \Psi^{-1}\left(h\left(-\tfrac{\partial}{\partial t}\cdot t\right)\cdot g\left(-\tfrac{\partial}{\partial t}\cdot t\right)\left[\tfrac{1}{f-t}\right]\right) \\ &= \Psi^{-1}\left(\left(h\left(-\tfrac{\partial}{\partial t}t\right)g\left(-\tfrac{\partial}{\partial t}t\right)\right)\cdot\left[\tfrac{1}{f-t}\right]\right) \\ &= \Psi^{-1}\left(hg\left(-\tfrac{\partial}{\partial t}t\right)\cdot\left[\tfrac{1}{f-t}\right]\right) \end{aligned}$$

$$= h(s)g(s)\cdot \underline{f}^s.$$

So $R_f[s]$-module on $R_f[s]\cdot \underline{f}^s$ is the same.

Now let $R = k[\underline{x}]$ poly ring.

(so $D_{R_f|k}[s] = R_f[s]\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle$).

$$\frac{\partial}{\partial x_i}\cdot \underline{f}^s = \psi^{-1}\left(\varphi\left(\frac{\partial}{\partial x_i}\right)\psi(\underline{f}^s)\right)$$

$$= \psi^{-1}\left(\frac{\partial}{\partial x_i}\cdot \left[\frac{1}{f-t}\right]\right)$$

$$= \boxed{\psi^{-1}\left(\left[\frac{-\frac{\partial f}{\partial x_i}}{(f-t)^2}\right]\right)}$$

$$\psi\left(\boxed{\frac{s\cdot \frac{\partial f}{\partial x_i}}{f}\cdot \underline{f}^s}\right) = \frac{\left(-\frac{\partial}{\partial t}\cdot t\right)\frac{\partial f}{\partial x_i}}{f}\left[\frac{1}{(f-t)}\right]$$

$$= \frac{-\frac{\partial}{\partial t}}{f}\cdot\left(\left[\frac{t\frac{\partial f}{\partial x_i}}{f-t}\right] + \underbrace{\left[\frac{(f-t)\left(\frac{\partial f}{\partial x_i}\right)}{f-t}\right]}_{0}\right)$$

$$= \frac{\frac{-\partial}{\partial t}}{f} \cdot \left(\left[\frac{f \frac{\partial f}{\partial x_i}}{f-t} \right] \right) = \frac{-\partial}{\partial t} \cdot \left[\frac{\frac{\partial f}{\partial x_i}}{f-t} \right]$$

$$= \left[\frac{-\frac{\partial f}{\partial x_i}}{(f-t)^2} \right].$$

So, $\frac{\partial}{\partial x_i} \cdot \underline{f}^s = \left(\frac{s}{f} \frac{\partial f}{\partial x_i} \right) \underline{f}^s$.

Conclusion: • $\exists$ $D_{R_f|K}[s]$-module structure on $R_f[s] \cdot \underline{f}^s$ as described earlier for $R = K[\underline{x}]$ (and thus a $D_{R|K}[s]$-module structure).

• Can define $R_f[s] \cdot \underline{f}^s$ even when R is not poly ring.

(This construction is important for other reasons.)

Bernstein-Sato polynomials

Def: The <u>Bernstein-Sato</u> functional equation is an equation of the form
$$P(s) \cdot f \underline{f}^s = b(s) \underline{f}^s$$
with $P(s) \in D_{R|K}[s]$, $b(s) \in K[s]$, as an equation in $R_f[s] \cdot \underline{f}^s$.

Note: We insist on $P(s) \in D_{R|K}[s]$, not $D_{R_f|K}[s]$, since in $D_{R_f|K}[s]$ have $1/f = P(s)$ and $b(s) = 1$. The point is to undo mult. of f without dividing by f.

$$P(s) \cdot f \underline{f}^s = b(s) \underline{f}^s \quad \text{in} \quad R_f[s] \cdot \underline{f}^s$$
$$\Updownarrow$$
$$P(t) \cdot f^{t+1} = b(t) f^t \quad \text{in } R_f \text{ for all } t \in \mathbb{Z}$$
($\Leftrightarrow$ infinitely many $t \in \mathbb{Z}$).

$R = k[x_1, \ldots, x_n]$.

Examples: 1) Let $x_i \in R$.

$$\frac{\partial}{\partial x_i} \cdot x_i^{t+1} = (t+1) x_i^t \quad \text{for all } t \in \mathbb{Z}.$$

Take $P(S) = \frac{\partial}{\partial x_i}$, $b(S) = S+1$

$$\Rightarrow \quad P(S) \cdot x_i \underline{x_i^S} = b(S) \cdot \underline{x_i^S} \quad \text{in } R_{x_i}[S] \cdot \underline{x_i^S}.$$

Also, $x_i \frac{\partial^2}{\partial x_i} \cdot x_i^{t+1} = (t+1)t \cdot x_i^t$ for all $t \in \mathbb{Z}$.

$P(S) = x_i \frac{\partial^2}{\partial x_i}$, $b(S) = S(S+1)$

yields functional equation.

2) Take $x_i^n \in R$.

$$\left(\frac{\partial}{\partial x_i}\right)^n \cdot x_i^{n(t+1)} = (nt+n)(nt+n-1)\cdots(nt+1)\, x_i^{nt}$$

$\rightsquigarrow$ $P(S) = \left(\frac{\partial}{\partial x_i}\right)^n$, $b(S) = (nS+n)(nS+n-1) \cdots (nS+1)$.

3) For $x_1^2 + x_2^3$, have

$$P(s) = \frac{1}{12} x_2 \frac{\partial^2}{\partial x_1} \frac{\partial}{\partial x_2} + \frac{1}{27} \frac{\partial^3}{\partial x_2} + \frac{s}{4} \frac{\partial}{\partial x_1} + \frac{3}{8} \frac{\partial^2}{\partial x_1}$$

$$b(s) = (s+1)(s+5/6)(s+7/6).$$

Next goal is to prove that every f admits a nonzero functional equation (poly ring char 0).

Prop: $R_f(s) \cdot \underline{f}^s$ is a holonomic $D_{R(s)|K(s)}$-module.

pf: Will give a filtration by $K(s)$ vector spaces that is consistent with the Bernstein filtration on $D_{R(s)|K(s)}$ that is small. If $f \in [R]_{\le a}$, set

$$F^t = \frac{1}{f^t} \cdot B^{(a+1)t} \cdot \underline{f}^s, \text{ where}$$

$B^\bullet$ is Ber. filtration on $D_{R(S)|K(S)}$.

Take $\frac{r}{f^t}\underline{f}^s \in F^t$, so $r \in B^{(a+1)t}$

- For $h(s) \in K(S)$, $h(s) \in B^0$

$$h(s)\frac{r}{f^t}\cdot \underline{f}^s = \frac{\overbrace{h(s)r}^{\in B^{(a+1)t}}}{f^t}\cdot \underline{f}^s \in F^t \quad ✓$$

- For $x_i \in B^1$

$$x_i \cdot \frac{r}{f^t}\cdot \underline{f}^s = \frac{\overbrace{\overbrace{x_i}^{\in B^1}\,\overbrace{f}^{\in B^a}\, r}^{\in B^{(a+1)(t+1)}}}{f^{t+1}}\underline{f}^s \in F^{t+1} \quad ✓$$

- For $\frac{\partial}{\partial x_i} \in B^1$

$$\frac{\partial}{\partial x_i}\cdot \frac{r}{f^t}\underline{f}^s = \frac{\frac{\partial}{\partial x_i}(r) f^t - t f^{t-1}\frac{\partial f}{\partial x_i}\cdot r}{f^{2t}}\cdot \underline{f}^s$$

$$= \frac{\overbrace{\overbrace{f}^{B^a}\overbrace{\left(\frac{\partial}{\partial x_i}\cdot r\right)}^{B^1}}^{B^{a+1}} + \left(-t\overbrace{\frac{\partial f}{\partial x_i}}^{B^a}\cdot r\right)}{f^{t+1}}\cdot \underline{f}^s$$

$$\in \frac{1}{f^{t+1}} B^{(a+1)(t+1)} \cdot f^s = F^{t+1}$$

Since $B^i = K(S) \cdot \left(\{x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}\right)^i$,

$F^\bullet$ is consistent with $B^\bullet$.

Easy to see $\dim_{K(S)} (F^t) \leq C t^n$ for some C.

$$\dim_{K(S)} (F^t) = \dim_{K(S)} \left(B^{(a+1)t}\right).$$

Thus, $R_f[S] \cdot f^s$ is holonomic. $\blacksquare$