Last time: For K field of char 0,
$R = K[\underline{x}]$ poly ring,
$H^i_I(R)$ has finite length as $D_{R|K}$-module
$\Rightarrow \mathrm{Ass}_R(H^i_I(R))$ is finite.

Ex: Let $R = \dfrac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$.

Then $H^1_{(x)}(R) = R_x/R$ does not have finite length as a $D_{R|\mathbb{C}}$-module:
$H^1_{(x)}(R)$ is graded, and has elements of arbitrarily low negative degree: $\left[\frac{1}{x^t}\right]$..

Note that $\delta \in [D_{R|\mathbb{C}}]_i$ acts on R_x as a map of degree i:

$$\delta(r/x^t) = \sum_{j=0}^{\operatorname{ord}(\delta)} \frac{\delta^{(j)}(r)}{(x^t)^{j+1}}, \text{ where } \delta^{(j)} = [\delta^{(j-1)}, x^t].$$

Then $\delta^{(j)}$ has degree $i+jt$, so

$$|\delta(r/x^t)| = |r| + i + jt - (tj+t)$$
$$= |r| + i - t$$
$$= |r/x^t| + i.$$

Likewise, $\delta \in [D_{R|C}]_i$ acts on $H^1_{(x)}(R)$ as a map of degree i.

But, $[D_{R|C}]_{<0} = 0$.

Thus, $H^1_{(x)}(R)$ is not finitely generated as a D-module.

(otherwise, look at lowest degree of an element in a generating set; anything generated by that set lives in larger degrees $\times\!\!\!\times$).

<u>Recall</u>: If $I = (f_1, \dots, f_t) \subseteq R$ (Noetherian commut. ring), then any element in $H^t_I(R)$ can be written as an equiv. class $\left[\frac{r}{(f_1 \cdots f_t)^k}\right]$, and

$$\left[\frac{r}{(f_1 \cdots f_t)^k}\right] = 0 \iff$$

$$\exists \ell : r(f_1 \cdots f_t)^\ell \in (f_1^{k+\ell}, \dots, f_t^{k+\ell}).$$

<u>Ex</u> (Singh): $R = \dfrac{\mathbb{Z}[u,v,w,x,y,z]}{(ux+vy+wz)}$.

Want to show that $H^3_{(u,v,w)}(R)$ has infinitely many associated primes.

<u>Key claim</u>: Let $\lambda_p = \dfrac{(ux)^p + (vy)^p + (wz)^p - (ux+vy+wz)^p}{p}$

(for each prime p)

$$= -\sum_{\substack{i+j+k=p \\ i,j,k \neq p}} \frac{\binom{p}{i,j,k}}{p} (ux)^i (vy)^j (wz)^k \in R.$$

$$\boxed{\binom{p}{i,j,k} = \frac{p!}{i!\,j!\,k!} \quad i+j+k=p}$$

The element $\mu_p = \left[\frac{\lambda_p}{u^p v^p w^p}\right]$ is nonzero in $H^3_{(u,v,w)}(R)$, but $p\mu_p = 0$ in $H^3_{(u,v,w)}(R)$.

$p\mu_p = 0$ since

$p\lambda_p = (ux)^p + (vy)^p + (wz)^p \in (u^p, v^p, w^p)$.

Need to check that

$$\boxed{\lambda_p (uvw)^\ell \notin (u^{p+\ell}, v^{p+\ell}, w^{p+\ell})R \quad \text{for } \underline{\text{all}}\ \ell \geq 0.} \quad (\ast)$$

Suppose otherwise. Can give R a $\mathbb{Z}^3$-grading

$|u| = (1,0,0)$ $\quad |v| = (0,1,0)$ $\quad |w| = (0,0,1)$

$|x| = (-1,0,0)$ $\quad |y| = (0,-1,0)$ $\quad |z| = (0,0,-1)$.

(ok, since $ux+vy+wz$ is homog.).

Then $|\lambda_p| = 0$, $|\lambda_p (uvw)^\ell| = (\ell,\ell,\ell)$.

$$\boxed{\lambda_p (uvw)^\ell = A\, u^{p+\ell} + B\, v^{p+\ell} + C\, w^{p+\ell}}$$

WLOG A, B, C homog $\Longrightarrow$

$$|A| = (-p, \ell, \ell) \Rightarrow A = x^p v^\ell w^\ell \cdot A'$$

$$|B| = (\ell, -p, \ell) \Rightarrow B = u^\ell y^p w^\ell \cdot B'$$

$$|C| = (\ell, \ell, -p). \Rightarrow C = u^\ell v^\ell z^p \cdot C'$$

with A', B', C' of degree $\underline{0}$.

$$\lambda_p \cancel{(uvw)^\ell} = A' \cancel{(uvw)^\ell} (ux)^p + B' \cancel{(uvw)^\ell} (vy)^p + C' \cancel{(uvw)^\ell} (wz)^p$$

$$\Rightarrow \boxed{\lambda_p \in ((ux)^p, (vy)^p, (wz)^p) R_{\underline{0}}}$$

where $R_{\underline{0}}$ = subring of R of degree $\underline{0}$ elts

$$= \frac{\mathbb{Z}[\underset{\alpha}{ux}, \underset{\beta}{vy}, \underset{\gamma}{wz}]}{\underbrace{(ux) + (vy) + (wz)}} \simeq \boxed{\mathbb{Z}[ux, vy]}$$

poly ring.

$$= \frac{\mathbb{Z}[\alpha, \beta, \gamma]}{(\alpha + \beta + \gamma)} = \mathbb{Z}[\alpha, \beta]$$

In $R_0 \simeq \mathbb{Z}[ux, vy]$,

$$\lambda_p = -\sum_{\substack{i+j+k=p \\ i,j,k \neq p}} \frac{\binom{p}{i,j,k}}{p} (ux)^i (vy)^j (-ux - vy)^k$$

the coefficient of $(ux)^{p-1}(vy)$ is nonzero mod p, so

$$\lambda_p \notin ((ux)^p, (vy)^p, (-ux-vy)^p, p)\mathbb{Z}[ux, vy]$$
$$= ((ux)^p, (vy)^p, p)\,\mathbb{Z}[ux, vy].$$

Thus, $\lambda_p \notin ((ux)^p, (vy)^p, (wz)^p) \dfrac{\mathbb{Z}[ux, vy, wz]}{(ux+vy+wz)}$.

$\divideontimes$.

Now, $R \cdot \mu_p$ is a nonzero submodule with annihilator $\ni p$, so it has an associated prime containing p.

Thus, $\exists\, Q_p \in \mathrm{Ass}_R(H^3_{(u,v,w)}(R))$ with $p \in Q_p$, for each prime p. These must be distinct (since $p, p' \in Q \Rightarrow 1 \in Q$ $\#$). $\boxed{}$

The D-module $R_f[s] \cdot f^s$

K field of char 0,

$R = K[\underline{x}]$ poly ring.

Write $R[s] := K[\underline{x}][s]$.

For $f \in R$, $R_f[s] := (R[s])_f$.

$R(s) := K(s)[\underline{x}] = (K[s] \setminus \{0\})^{-1} R[s]$.

$R_f(s) := (R(s))_f$.

$$D_{R|K}[S] := D_{R|K} \otimes_K K[S]$$
$$= \bigoplus_{\beta} \overline{K[\underline{x}, S]} \frac{\partial}{\partial x_1}^{\beta_1} \cdots \frac{\partial}{\partial x_n}^{\beta_n}$$
$$= R[S]\left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle.$$

$$D_{R|K}(S) := D_{R|K} \otimes_K K(S)$$
$$= (K[S] \setminus \{0\})^{-1} D_{R|K}[S]$$
$$= \bigoplus_{\beta} \overline{K(S)[\underline{x}]} \frac{\partial}{\partial x_1}^{\beta_1} \cdots \frac{\partial}{\partial x_n}^{\beta_n}$$
$$= D_{R(S)|K(S)}$$
$$= R(S)\left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle.$$

<u>Warning</u>: Can't always localize by mult. closed set in noncommutative ring, but can do this for mult. closed subset of the center of ring.

We define $R_f[S]\cdot \underline{f}^s$ as:

- as an $R[S]$-module, this is the module $R_f[S]$ (with $\underline{f}^s$ as formal generator).
- as a $D_{R|K}[S]$-module,

$$\frac{\partial}{\partial x_i}\cdot g\underline{f}^s = \left(\frac{\partial g}{\partial x_i} + \frac{sg}{f}\frac{\partial f}{\partial x_i}\right)\underline{f}^s$$

Likewise, $R_f(S)\cdot \underline{f}^s$ is a $D_{R|K}(S)$-module by the same rules.

Rmk: One needs to check that these yield a consistent structure of $D_{R|K}[S]$ or $D_{R|K}(S)$-module; we leave it as an exercise for now.

Prop: For each $t \in \mathbb{Z}$, the map

$$R_f[s]\,\underline{f}^s \xrightarrow{\pi_t} R_f$$

given by $\pi_t(g(s)\cdot \underline{f}^s) = g(t) f^t$ is a homomorphism of $D_{R|K}$-modules.

Pf: Suffices to check that

$$\pi_t(\alpha \cdot g(s)\underline{f}^s) = \alpha\, \pi_t(g(s)\underline{f}^s)$$

for all generators α of $D_{R|K}$ and all $g(s) \in R_f[s]$. Namely, α is $\overline{x_i}$ or $\frac{\partial}{\partial x_i}$.

$$\pi_t(\overline{x_i}\, g(s)\underline{f}^s) = x_i\, g(t) f^t = \overline{x_i}\, \pi_t(g(s)\underline{f}^s).$$

$$\pi_t\left(\frac{\partial}{\partial x_i}\, g(s)\underline{f}^s\right) = \pi_t\left(\left(\frac{\partial g(s)}{\partial x_i} + \frac{s\, g(s)}{f}\frac{\partial f}{\partial x_i}\right)\cdot \underline{f}^s\right)$$

$$= \left(\frac{\partial g(t)}{\partial x_i} + \frac{t\, g(t)}{f}\frac{\partial f}{\partial x_i}\right) f^t$$

$$= \frac{\partial g(t)}{\partial x_i} f^t + t\, g(t) \frac{\partial f}{\partial x_i} f^{t-1}$$

$$= \frac{\partial}{\partial x_i}\left(g(t)\cdot f^t\right) = \frac{\partial}{\partial x_i} \pi_t(g(s) \underline{f}^s) . \quad \blacksquare$$

We also note:

Prop: For $g(s)\underline{f}^s \in R_f[s]\underline{f}^s$

$$g(s)\underline{f}^s = 0 \iff \pi_t(g(s)\underline{f}^s) = 0$$

for all $t \in \mathbb{Z}$

($\iff$ for infinitely many $t \in \mathbb{Z}$).

Pf: Write $g(s) = \dfrac{g_b s^b + \cdots + g_0}{f^a}$

with $g_i \in R$.

So, $\pi_t(g(s)\underline{f}^s) = \dfrac{g_b t^b + \cdots + g_0}{f^a} \cdot f^t$.

and $\gamma_t(g(s)\, f^s) = 0$

$\iff$ $s = t$ is a root of ~~of~~ $g_b s^b + \cdots + g_0$ in $\mathrm{frac}(R)$.

(Since poly in one var. with ∞ roots $\iff$ is 0 poly). $\blacksquare$