$K$ field of char $0$

$R = K[x_1, \dots, x_n]$

$M$ f.g $D_{R|K}$-module

$\Rightarrow$ $gr(M, \mathcal{G}^\bullet)$ f.g module. as a $\underline{gr^{Ber}(D_{R|K})}_{\cong K[x_1, \dots, x_n, y_1, \dots, y_n]}$-module

for $\mathcal{G}^\bullet$ good compat. with Ber.

$\rightsquigarrow$ Consider, $V(\operatorname{ann}_{gr^{Ber}(D_{R|K})}(gr(M, \mathcal{G}^\bullet)))$

the <u>support variety</u> of $M$.

<u>Exercise:</u> The support variety of a f.g. $D_{R|K}$-module is independent of the choice of $\mathcal{G}^\bullet$, but $\operatorname{ann}_{gr^{Ber}(D_{R|K})}(gr(M, \mathcal{G}^\bullet))$ is not.

$\dim(M) = \dim(\text{supp variety of } M)$.

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Recall: A holonomic D-module is f.g. D-module of dim $n$ (where $R = K[x_1, \dots, x_n]$, $K$ field char $0$).

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Prop: $K$ field char $0$, $R = K[x_1, \dots, x_n]$, $M$ D-module (not a priori f.g.). If there is a filtration $F^\bullet$ on $M$ compatible with Ber st. $\exists C > 0$:

$$\dim_K (F^t) \leq C t^n, \text{ for all } t \gg 0,$$

then $M$ is fin. gen., and hence holonomic.

Pf: Want to show that $M$ is a Noetherian $D$-module, which implies f.g. It suffices to show that every chain of f.g. submodules stabilizes.

If $\overset{\neq 0}{L} \subseteq M$ is f.g., then $F^\bullet \cap L$ is a filtration on $L$ of dim $\leq n$ since $\dim_K(F^t \cap L) \leq \dim_K(F^t)$.
By Bernstein's inequality, $\dim(L) = n$.
Then, $e_n(F^\bullet \cap L) \leq e_n(F^\bullet) \leq n!\,C$.
By Lemma from last time,

$e(L) \leq e_n(F^\bullet \cap L)$, so $e(L) \leq n!\,C$.

Thus, if $0 \subsetneq L_1 \subsetneq L_2 \subsetneq L_3 \subsetneq \cdots$ is a chain of f.g. submodules (proper),

we have $e(L_i) \gneq e(L_{i-1})$ each $i$,
so $e(L_i) \geq i$, but $e(L_i) \leq n!C$,
so the chain cannot consist of
$n!C$ modules. $\blacksquare$

<u>Rmk</u>: In the context of last prop,
suffices to show $\dim_K(F^t)$ is
bounded by a polynomial of degree $n$.

<u>Prop</u>: $K$ field of char $0$, $R = K[x_1, \dots, x_n]$.
Let $M$ be holonomic. Then for any $f \in R$,
$M_f$ is holonomic.
<u>Pf</u>: Let $G^\bullet$ be a good filtration
on $M$. Let $f$ have degree
at most $a$ ($f \in [R]_{\leq a}$).

Then $\bar{f} \in B^a$.

Set $F^t := \frac{1}{f^t} \cdot \sigma^{(a+1)t}$.

If $\frac{m}{f^t} \in F^t$, so $m \in \sigma^{(a+1)t}$,

then $\bar{x}_i \cdot \frac{m}{f^t} = \frac{x_i \cdot m}{f^t} = \frac{\bar{x}_i f \cdot m}{f^{t+1}}$,

with $\underbrace{\bar{x}_i}_{B^1}\underbrace{f}_{B^a} \cdot \underbrace{m}_{\sigma^{(a+1)t}} \in B^{a+1}\sigma^{(a+1)t} \subseteq \sigma^{(a+1)(t+1)}$

$\Rightarrow \bar{x}_i \cdot \frac{m}{f^t} \in F^{t+1}$

Likewise, $\frac{\partial}{\partial x_i} \cdot \frac{m}{f^t} = \frac{\overbrace{f}^{B^a}\left(\overbrace{\frac{\partial}{\partial x_i}}^{B^1} \cdot \overbrace{m}^{\sigma^{(a+1)t}}\right) - \overbrace{\left(\frac{\partial}{\partial x_i}(f)\right)}^{B^a} \cdot \overbrace{m}^{\sigma^{(a+1)t}}}{f^{t+1}}$

$\in \frac{1}{f^{t+1}} \sigma^{(a+1)(t+1)} = F^{t+1}$.

Thus, $B^1 \cdot F^t \subseteq F^{t+1}$.

Since $B^s = \underbrace{B^1 \cdots B^1}_{s}$, get that

$F^\bullet$ is compatible with Ber.

Then, $\dim_K(F^t) = \dim_K\left(\frac{1}{f^t}\cdot G^{(a+1)t}\right)$

$$= \dim_K\left(G^{(a+1)t}\right)$$

$$= \frac{e(M)}{n!}\left((a+1)t\right)^n + \text{lower order terms}$$

$$= \frac{e(M)(a+1)^n}{n!} t^n + \text{L.O.T.}$$

Then, by last Proposition, $M_f$ is holonomic. ∎

Thm: Let $K$ field of char $0$, $R = K[x_1, \dots, x_n]$, $M$ holonomic $D$-module. For any ideal $I \subseteq R$, $H^i_I(M)$ is holonomic. In particular, $H^i_I(R)$ is holonomic.

Pf: The Čech complex on a gen. set for $I$ is a complex of holonomic $D$-modules. Since submods & quot. mods of holo. mods. are holo, cohomology of Čech complex is holo. $\boxed{}$

Ex: Localization at an arbitrary mult. set is not necessarily holo. Moreover, is not necessarily f.g. as $D$-module. If $R = \mathbb{C}[x]$, $M = \mathbb{C}(x)$, then $M$ is not fin.gen.; otherwise write $M = D_{R|\mathbb{C}} \cdot \langle \frac{r_1}{s_1}, \dots, \frac{r_t}{s_t} \rangle$.

Have $\frac{r_1}{s_1}, \dots, \frac{r_t}{s_t} \in R_{s_1 \cdots s_t}$, which is a $D_{R|K}$-module, so

$$D_{R|K} \cdot \langle \frac{r_1}{s_1}, \dots, \frac{r_t}{s_t} \rangle \subseteq R_{s_1 \cdots s_t} \text{ ※}.$$

The same argument shows that only D-mod f.g. localizations are localizations of form $R_f$.

<u>Prop</u>: Let $A \to R$ rings, R Noetherian, $M \neq 0$ a $D_{R|A}$-mod.
<u>If</u> $M$ is simple, then $\mathrm{Ass}_R(M)$ is a singleton.

<u>prf</u>: For $P \in \mathrm{Ass}_R(M)$, $H^0_P(M)$ is nonzero, since $R/p \hookrightarrow M$, and

$$H^0_p(M) = \ker(M \to \bigoplus_i M_{f_i}) =$$

$$= \{m \in M \mid \exists t : p^t m = 0\},$$

so image of $R/p$ in $M$ is contained in $H^0_p(M)$.

Since $M$ is simple and $H^0_P(M)$ is a $D_{R|A}$-submodule, $H^0_P(M) = M$.

Then, if $P, Q \in \mathrm{Ass}_R(M)$, $\exists m \in M$ with $\mathrm{ann}_R(m) = Q$, so $\exists n$:

$P^n \subseteq \mathrm{ann}_R(m) = Q$, so $P \subseteq Q$.

Likewise $Q \subseteq P$, so $P = Q$. $\boxtimes$

<u>Thm [Lyubeznik]</u>: $K$ field of char $0$, $R = K[x_1, \dots, x_n]$, then any holonomic $D$-module $M$ has finitely many assoc. primes as an $R$-module.

In particular, any $H^c_I(R)$ has finitely many assoc. primes.

(composition series)

Pf: Take a filtration of $M$ by simple $D$-modules (can do since $\ell_{D_{R|K}}(M) < \infty$):

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\ell = M.$$

Have $\operatorname{Ass}_R(M_i) \subseteq \operatorname{Ass}(M_{i-1}) \cup \underbrace{\operatorname{Ass}(M_i/M_{i-1})}_{\text{each is a singleton.}}$

By induction on $i$, each $M_i$, in particular $M_\ell = M$, has finitely many associated primes. $\square$

This is not true in general for Noetherian rings!

Ex [Singh]: Let $R = \frac{\mathbb{Z}[u,v,w,x,y,z]}{(ux+vy+wz)}$.

Will show that $H^3_{(u,v,w)}(R)$ has infinitely many associated primes.

We are looking at

$$\overset{2}{R_{uv} \oplus R_{vw} \oplus R_{uw}} \to \overset{3}{\underbrace{R_{uvw}}_{\text{cohomology here}}} \to \overset{4}{0}$$

So $H^3_{(u,v,w)}(R) = \frac{R_{uvw}}{\mathrm{im}(R_{uv} \oplus R_{uw} \oplus R_{vw})}$.

Can write any element as

$\left[\frac{r}{(uvw)^t}\right]$ some $r \in R$, $t \in \mathbb{N}$.

we have $\left[\frac{r}{(uvw)^t}\right] = 0 \iff \frac{r}{(uvw)^t} = \frac{r_1}{(uv)^a} + \frac{r_2}{(uw)^b} + \frac{r_3}{(vw)^c}$

$\exists\, r_1, r_2, r_3 \in R$
$\exists\, a, b, c \in N$

$$\iff \frac{r}{(uvw)^t} = \frac{r_1}{(uv)^{t+k}} + \frac{r_2}{(uw)^{t+k}} + \frac{r_3}{(vw)^{t+k}}$$

$\exists\, r_1, r_2, r_3 \in R$
$\exists\, k \in N$

$$\iff r(uvw)^k = r_1 w^{t+k} + r_2 v^{t+k} + r_3 u^{t+k}$$

$\exists\, r_1, r_2, r_3 \in R$
$\exists\, k \in N$

$$\iff \exists\, k \in N : r(uvw)^k \in (u^{t+k}, v^{t+k}, w^{t+k})$$