$K$ field of char $0$, $R = K[x_1, \dots, x_n]$ poly.

$D = D_{R|K}$

For a f.g. $D$-module $M$, define

$d(M) :=$ dimension of $G^\bullet$ good filtration on $M$

$= \dim_{gr(D, B^\bullet)} (gr(M, G^\bullet))$

standard graded poly ring.

Bernstein filtration

where $G^\bullet$ compatible with $B^\bullet$ & $gr(M, G^\bullet)$ f.g. $gr(D, B^\bullet)$-module.

Independent of choice of $G^\bullet$.

$e(M) :=$ multiplicity of $gr(M, G^\bullet)$ as a $gr(D, B^\bullet)$-module.

also independent of $G^\bullet$

Ex: Take $M = H^n_{(\underline{x})}(R) \simeq D/D\cdot\langle \underline{x} \rangle$.

$M \simeq x_1^{-1}\cdots x_n^{-1} K[x_1^{-1},\dots,x_n^{-1}]$ as graded $K$-vectorspaces.

This is generated by $\mu = \left[\frac{1}{x_1\cdots x_n}\right]$

Then, $G^t = B^t \cdot \mu$ is a good filtration.

$$\begin{aligned} &\bigoplus_{|a|+|b|\le t} K\cdot \underline{x}^a \partial^{(b)} \cdot \mu \\ &= \bigoplus_{|a|+|b|\le t} K \cdot \left[\frac{1}{x_1^{1+\beta_1-\alpha_1}\cdots x_n^{1+\beta_n-\alpha_n}}\right] \\ &= [M]_{\ge -n-t}. \end{aligned}$$

Have $\dim_K([M]_{\ge -n-t}) = \dim_K([R]_{\le t})$

$$\Rightarrow d(H^d_{(\underline{x})}(R)) = d(R) = n$$

$$e(H^d_{(\underline{x})}(R)) = e(R) = 1.$$

Exercise: Compute $d, e$, of $R_{x_1}$ using the definitions. (Will see $e(R_x) \neq 1$).

(in char 0 poly setting).

**Lem:** If $M$ is a fin. gen. $D$-module with any filtration $F^\bullet$ compatible with Ber., then $\dim(M, F^\bullet) \geq d(M)$, and if $\dim(M, F^\bullet) = d(M)$, then $e_{d(M)}(M, F^\bullet) \geq e(M)$.

Pf: Boils down any good filtration is contained in a (uniform) shift of an arbitrary filtration. (Exercise). ☒

**Prop:** In char 0 poly ring setting, let $0 \to L \to M \to N \to 0$ be a short exact sequence of fin. gen. $D$-modules. Then
- $d(M) = \max\{d(L), d(N)\}$,

and • if $d(L)=d(N)$, then

$$e(M)=e(L)+e(N).$$

pf: Take $G^\bullet$ good filtration on $M$ w.r.t. Ber. Claim: $G^\bullet \cap L$ is a good filtration on $L$, and $G^\bullet N$ is a good filt. on $N$.

Have seen that there is SES

$$0 \to gr(L, G^\bullet \cap L) \to gr(M, G^\bullet) \to gr(N, G^\bullet N) \to 0$$

of $gr(D_{R|k}, B^\bullet)$. Since $gr(D_{R|k}, B^\bullet)$ is Noeth and $gr(M, G^\bullet)$ is f.g., so is $gr(L, G^\bullet \cap L)$, so $G^\bullet \cap L$ is good. $gr(N, G^\bullet N)$ is quotient of f.g., so f.g. Claim ✓.

Then, $\underset{d(M)}{\underset{\shortparallel}{\dim(gr(M,G^\bullet))}} = \max\{\underset{d(L)}{\underset{\shortparallel}{\dim(gr(L,G^\bullet\cap L))}}, \underset{d(N)}{\underset{\shortparallel}{\dim(gr(N,G^\bullet N))}}\}$, and likewise for $e(L), e(M), e(N)$. $\boxtimes$

__Lem:__ Let $\delta \in B^t$. Then $[\delta, x_j], [\delta, \frac{\partial}{\partial x_j}] \in B^{t-1}$.

__pf:__ Exercise.

__Th'm [Bernstein's inequality]:__ $K$ field of char $0$, $R = K[\underline{x}]$ poly ring of dim $n$, $(M, G^\circ)$ is a $(D_{R|K}, B^\bullet)$-module. If $M \neq 0$, then $\dim(M, G^\bullet) \geq n$. Thus, if $M$ is fin gen, $d(M) \in \{n, n+1, \dots, 2n\}$.

__Pf:__ Show by induction on $t$ that the map of vector spaces

$$\begin{aligned} B^t &\longrightarrow \operatorname{Hom}_K(G^t, G^{2t}) \\ \delta &\longmapsto (m \mapsto \delta m) \end{aligned}$$

is injective.

Need to see that $\delta \in B^t \setminus \{0\}$, then $\delta(\mathcal{G}^t) \neq 0$.

For $t=0$, $B^0 = K$, so $B^0 \setminus \{0\} = K^\times$ ✓.

Inductive step:

If $[\delta, \overline{x_i}] = 0$ for all $i$, then

$$\delta = \overline{r} \in \overline{R \cap B^t}$$

$$\Rightarrow \left[\delta, \frac{\partial}{\partial x_i}\right] = -\frac{\partial}{\partial x_i}(r) \neq 0 \text{ some } i$$

unless $\overline{r} \in B^0$ (in which case we're done).

Can assume that either $[\delta, \overline{x_i}] \neq 0$ $\in B^{t-2}$

or $\left[\delta, \frac{\partial}{\partial x_i}\right] \neq 0$ $\in B^{t-2}$ some $i$.

$$(\delta \overline{x_i} - \overline{x_i}\delta)(\mathcal{G}^{t-1}) \overset{IH}{\neq} 0 \qquad \left(\delta\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}\delta\right)(\mathcal{G}^{t-1}) \neq 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$0 \neq \delta(\overline{x_i}\mathcal{G}^{t-1}) \subseteq \delta(\mathcal{G}^t) \qquad 0 \neq \delta\left(\frac{\partial}{\partial x_i}\mathcal{G}^{t-1}\right) \subseteq \delta(\mathcal{G}^t)$$

— or —      — or —

$$0 \neq \delta(\mathcal{G}^{t-1}) \subseteq \delta(\mathcal{G}^t) \qquad 0 \neq \delta(\mathcal{G}^{t-1}) \subseteq \delta(\mathcal{G}^t).$$

This completes the claim.

Thus,

$$\binom{2n+t}{t} = \dim_K(B^t) \le \dim_K\left(\mathrm{Hom}_K(G^t, G^{2t})\right)$$

$$= \dim_K(G^t)\cdot\dim_K(G^{2t})$$

and $\binom{2n+t}{t} = \dfrac{t^{2n}}{(2n)!}$ + lower order terms.

$$\Rightarrow \limsup_t \frac{\log\left(t^{2n}/(2n)!\right)}{\log(t)} \le \limsup_t \frac{\log\left(\dim(G^t)\dim(G^{2t})\right)}{\log(t)}$$

$$2n = \limsup_t \frac{\boxed{\log(\dim(G^t))} + \log(\dim(G^{2t}))}{\log(t)}$$

But, $\displaystyle \limsup \frac{\log(\dim(G^{2t}))}{\log(t)}$

$$= \limsup \frac{\log(\dim(G^{2t}))}{\boxed{\log(t)+\log(2)}} \quad \log(2t)$$

$$\leq \limsup_t \frac{\log(\dim(\mathcal{G}^t))}{\log(t)} = \dim(M, \mathcal{G}^\bullet)$$

Thus, $2\dim(M, \mathcal{G}^\bullet) \geq 2n$.

$$\dim(M, \mathcal{G}^\bullet) \geq n. \qquad \blacksquare$$

Def: $K$ field of char $0$, $R = K[x_1, \ldots, x_n]$.
A $D$-module is holonomic if it is finitely generated and $d(M) = n$, or else $M = 0$.

We will say $e(0) = 0$. If $M$ is holonomic, then $M = 0 \Leftrightarrow e(0) = 0$.
Then, if $0 \to L \to M \to N \to 0$ are holonomic, $e(M) = e(L) + e(N)$.

Ex: Some Holonomic modules are:

$\mathcal{O}$, $R$, $H^n_{(\underline{x})}(R)$.

A nonholonomic module is $D_{R|K}$. (if $n>0$)

Rmk: Submodules & quotient modules of holonomic modules are holonomic:

if $N \subseteq M$, $d(N) \leq d(M) = n$,

so by Bernstein, $N=0$ or $d(N)=n$.

$N$ is fin. gen., since $M$ is fin. gen., and $D_{R|K}$ is left Noetherian.

(if $T$ is left Noeth, then f.g. $\Longleftrightarrow$ Noeth for left-$T$-modules)

Prop: If $M$ is a holonomic $D$-module, then $M$ has finite length as a $D$-module; moreover,

$$\ell_{D_{R|K}}(M) \leq e(M).$$

Pf: If $0 \to L \to M \to N \to 0$ SES of holonomic $D$-modules, and $e(L) = e(M)$, then $e(N) = 0 \Rightarrow N = 0$, $\Rightarrow L = M$. Thus, given a chain of submodules (proper)

$$\cdots \subsetneq M_2 \subsetneq M_1 \subsetneq M$$

Each $M_i$ is necessarily holonomic.

We have $e(M) \gneq e(M_1) \gneq e(M_2) \gneq \cdots$

Since these are all positive integers, the chain must have length at most $e(M)$. ∎

Ex: Since $R$, $H^n_{(x)}(R)$ are holonomic with $e=1$, they have length at most 1, so they are simple $D$-modules.

In general "=" is rare.

"=" $\Rightarrow$ every composition factor has multiplicity 1.

Q: What can we say about holonomic $D$-modules of multiplicity 1 $\Longleftrightarrow$ holo $D$-mods with $\ell = e$?