Q: Over $(T, F^\bullet)$ filtered k-alg. with $\mathrm{gr}(T, F^\bullet)$ commut., f.g. k-alg,
M left T-module with good filtration $G^\bullet$.
Is any filtration $G^\bullet$ on M good?

A: No, e.g.,
$T = D_{\mathbb{C}[x]|\mathbb{C}}$ with $F^\bullet$ order filtration.
$M = H^1_{(x)}(\mathbb{C}[x])$.
$M = D_{\mathbb{C}[x]|\mathbb{C}} \cdot \left[\frac{1}{x}\right]$, a good filtration
is $G^i = T^i \cdot \left[\frac{1}{x}\right] = D^i_{\mathbb{C}[x]|\mathbb{C}} \cdot \left[\frac{1}{x}\right]$

$$= \bigoplus_{j \le i} \mathbb{C}[x] \left(\frac{\partial}{\partial x}\right)^j \cdot \left[\frac{1}{x}\right]$$

$$= \bigoplus_{a \le i+1} \mathbb{C} \left[\frac{1}{x^a}\right] \doteq [M]_{\ge -i-1}$$

$\boxed{\dim_k (G^i) = i+1}$

Filtration condition for $H^\bullet$:
$F^i H^j \subseteq H^{i+j}$

can preserve this containment by making $H^\bullet$ bigger & bigger...

Set $H^i := G^{\lfloor \sqrt{2} i \rfloor} = \bigoplus_{a \le \lfloor \sqrt{2} \cdot i \rfloor + 1} \mathbb{C}\left[\frac{1}{x^a}\right]$

$$= [M]_{\geq -\sqrt{2} i - 1}.$$ $\dim_k (H^i)_{\lfloor \sqrt{2} i \rfloor + 1}$

$$D^i_{\mathbb{C}[x]|\mathbb{C}} \cdot H^j = D^i_{\mathbb{C}[x]|\mathbb{C}} \cdot G^{\lfloor \sqrt{2} j \rfloor}$$
$$\subseteq G^{\lfloor \sqrt{2} j + i \rfloor} \subseteq G^{\lfloor \sqrt{2}(i+j) \rfloor} = H^{i+j}.$$

Exercise: $gr(M, H^\bullet)$ is not a finitely generated $gr^{ord}(D_{\mathbb{C}[x]})$-module.

Bernstein filtration

Want to consider smaller filtration on $D_{K[x]|K}$, K field of char 0.

Def: For K field of char 0, $R = K[\underline{x}]$, on $D_{R|K}$, we set

$$B^i = K\cdot\left\{ \begin{array}{l}\delta \in D_{R|K} \text{ homogeneous} \\ 2\,\mathrm{ord}(\delta) + \deg(\delta) \le i \end{array}\right\}.$$

$B^\bullet$ is called the <u>Bernstein filtration</u>.

We need to check that this is a filtration:

On monomial basis $\underline{x}^a \frac{\partial^{b_1}}{\partial x_1} \cdots \frac{\partial^{b_n}}{\partial x_n} = \mu_{a,b}$

we have $2\,\mathrm{ord}(\mu_{a,b}) + \deg(\mu_{a,b})$

$$= 2|b| + |a| - |b| = |a| + |b|.$$

Thus, $B^i \subseteq B^{i+1}$ and $\bigcup_i B^i = D_{R|K}$.

If $\alpha \in B^i$, $\beta \in B^j$ homogeneous, then

$\deg(\alpha\beta) = \deg(\alpha) + \deg(\beta)$

$\mathrm{ord}(\alpha\beta) \le \mathrm{ord}(\alpha) + \mathrm{ord}(\beta)$, so

$\deg(\alpha\beta) + 2\,\mathrm{ord}(\alpha\beta) \le i + j$.

Thus, $B^\bullet$ is multipliative.

Concretely, $B^i = \bigoplus_{|a|+|b|\le i} K \cdot \bar{x}^a \frac{\partial^{b_1}}{\partial x_1} \cdots \frac{\partial^{b_n}}{\partial x_n}$.

Note that each B^i is a fin. dim. K-vectorspace, whereas for D° order filtration, each D^i is a finite rank free R-module.

Compute associated graded:

$$B^i / B^{i-1} \simeq \bigoplus_{|a|+|b|=i} K \cdot \left(\bar{x}^a \frac{\partial^{b_1}}{\partial x_1} \cdots \frac{\partial^{b_n}}{\partial x_n} + B^{i-1}\right).$$

Note that $(\bar{x}_i + B^0)(\frac{\partial}{\partial x_i} + B^0)$

$$= \bar{x}_i \frac{\partial}{\partial x_i} + B^1 = \left(\frac{\partial}{\partial x_i} \bar{x}_i - 1\right) + B^1$$

$$= \frac{\partial}{\partial x_i} \bar{x}_i + B^1 = \left(\frac{\partial}{\partial x_i} + B^0\right)(\bar{x}_i + B^0).$$

Likewise, since $\bar{x}_i$ & $\frac{\partial}{\partial x_j}$, or $\bar{x}_i$ & $\bar{x}_j$, or $\frac{\partial}{\partial x_i}$ & $\frac{\partial}{\partial x_j}$ for $i \neq j$ commute in $D_{R|K}$, their images commute in $\mathrm{gr}(D_{R|K}, B^{\bullet})$.

This gives a map of K-algebras

$$K[y_1, \dots, y_n, z_1, \dots, z_n] \longrightarrow \mathrm{gr}(D_{R|K}, B^{\bullet})$$
$$y_i \longmapsto x_i + B^0$$
$$z_i \longmapsto \frac{\partial}{\partial x_i} + B^0.$$

Have $y_1^{a_1} \cdots y_n^{a_n} z_1^{b_1} \cdots z_n^{b_n} \longmapsto x^a \frac{\partial^{b_1}}{\partial x_1} \cdots \frac{\partial^{b_n}}{\partial x_n} + B^{|a|+|b|-1}$,

~~so~~ the homomorphism is a surjection, is a map of graded K-algebras.

Have same vector space dimension in each graded piece, so this is an isomorphism.

Thus, $\mathrm{gr}^{Ber}(D_{R|K}) := \mathrm{gr}(D_{R|K}, B^{\bullet})$ is a standard graded poly ring in $2n$ variables.

Dimension & multiplicity for D-modules

Let $R = k[\underline{x}]$ poly ring, k field of char 0.

Let M be a f.g. D-module.

Then M admits a good filtration compatible with the Bernstein filtration, say $G^\bullet$. Note that $G^\bullet$ is not unique.

E.g., if $M = D \cdot \{m_1, \dots, m_t\}$

$\rightsquigarrow G^\bullet = B^\bullet\{m_1, \dots, m_t\}$ good

Take some $m' \in G^1$.

$\rightsquigarrow G'^\bullet = B^\bullet\{m_1, \dots, m_t, m'\}$ also good

with $m' \in (G')_0$.

$\mathrm{gr}(M, G^\bullet)$ is a f.g. graded $\mathrm{gr}^{Ber}(D_{R|k})$-module.

Can use theory of Hilbert functions:

$H(gr(M,\sigma^\bullet), n) = \dim_k([gr(M,\sigma^\bullet)]_{\leq n})$

is a polynomial function for $n \geq t$ for some t.

The degree of this poly is the dimension of $gr(M,\sigma^\bullet)$ as a $gr^{Ber}(D_{R|k})$-module, call it d, and $d!$ times leading coefficient is a positive integer.

<u>Def:</u> For a filtered module $(M,\sigma^\bullet)$ with each σ^i a ~~fin.~~ fin. dim. k-vectorspace, we define $\dim(M,\sigma^\bullet) := \limsup_{n\to\infty} \frac{\log(\dim_k(\sigma^n))}{\log(n)}$.

and for an integer d, $e_d(M,\sigma^\bullet) := \limsup_{n\to\infty} \frac{d!\,\dim_k(\sigma^n)}{n^d}$.

Ex: In notation of beginning example, take $H^i = G^{(i^2)}$.

$\dim_k(G^i) = i+1 \rightsquigarrow \dim(G^\cdot) = 1$

$\dim_k(H^i) = i^2+1 \rightsquigarrow \dim(H^\cdot) = 2$

Take $J^i = G^{(e^i)} \rightsquigarrow \dim(J^\cdot) = \limsup_{n\to\infty} \frac{\log(e^n+1)}{\log(n)}$

$= \limsup_{n\to\infty} \frac{n}{\log(n)} = \infty.$

Prop: $R = K[\underline{x}]$, k field of char 0. Let M be a f.g. D-module with $G^\cdot$ good filtration w.r.t. Bernstein filt. Then $\dim(M, G^\cdot) \in \{0, 1, \ldots, 2n\}$, and for $d = \dim(M, G^\cdot)$, $e_d(M, G^\cdot)$ is a positive integer.

Pf: Follows from Hilbert function

discussion: $\dim_K(G^n)$

$$= \sum_{i=0}^{n} \dim_K(G^i/G^{i-1}) \quad (\text{where } G^{-1}=0)$$

(via SES's $0 \to G^{i-1} \to G^i \to G^i/G^{i-1} \to 0$)

$$= \sum_{i=0}^{n} \dim_K([gr(M,G^\bullet)]_i)$$

$$= \dim_K([gr(M,G^\bullet)]_{\leq n}) = H(gr(M,G^\bullet), n).$$

So $\dim_K(G^n) = a n^d +$ lower order (for $n \geq t$ some t)

with $d!a \in \mathbb{N}_{>0}$.

Then, $\dim(M,G^\bullet) = \limsup\limits_{n\to\infty} \dfrac{\log(an^d)}{\log(n)} = \dfrac{d\log(n)+\log(a)}{\log(n)} \to d.$

and $e_d(M,G^\bullet) = d!a \in \mathbb{N}_{>0}$. ☐

<u>Thm</u>: K field char 0, $R=K[\underline{x}]$ poly, M f.g. D-module. ~~If~~ $G^\bullet, H^\bullet$ are

good filtrations on $M^\bullet$ w.r.t. Bernstein filtration, then $\dim(M, G^\bullet) = \dim(M, H^\bullet)$, and if we call this value d, then

$$e_d(M, G^\bullet) = e_d(M, H^\bullet).$$

Pf: $\exists c$ s.t. $G^{n-c} \subseteq H^n \subseteq G^{n+c}$ for all n, ~~so~~

$$\dim_K(G^{n-c}) \le \dim_K(H^n) \le \dim_K(G^{n+c}).$$

Write $\dim(M, G^\bullet) =: d_G$, $e_{d_G}(M, G^\bullet) =: e_G$ and likewise for $H^\bullet$.

Since $\dim_K(G^n) = \frac{e_G}{d_G!} n^{d_G}$ + lower order terms,

have $\dim_K(G^{n+c}) = \frac{e_G}{d_G!}(n+c)^{d_G} +$ ——— " ———

$$= \frac{e_G}{d_G!}\big(n^{d_G} + \underbrace{\tbinom{d_G}{1} n^{d_G-1} c + \cdots}\big) + \text{——— " ———}$$

$$= \frac{e_G}{d_G!} n^{d_G} + \text{——— " ———}$$

same for $-c$.

$e_G/d_G!$
$\shortparallel$

$$\lim \frac{\dim_k(G^n)}{n^{d_G}} \le \lim \frac{\dim_k(G^{n-c})}{n^{d_G}} \le \lim \frac{\dim_k(H^n)}{n^{d_G}}$$

$$\le \lim \frac{\dim_k(G^{n+c})}{n^{d_G}} \le \lim \frac{\dim_k(G^n)}{n^{d_G}} = \frac{e_G}{d_G!},$$

So $\lim_n \frac{\dim_k(H^n)}{n^{d_G}}$ is a (finite) positive integer.

Thus, the degree of $\dim_k(H^n)$ (as a polynomial for $n \gg 0$) is d_G, and $e_{d_H}(M, H^\circ) = e_G$. $\boxed{\text{Thm}}$

<u>Def:</u> K field char 0, $R = K[\underline{x}]$, M finitely generated D-module. Then we define

$d(M) := \dim(M, G^\circ)$

$e(M) := e_d(M, G^\circ)$ for a good filtration G°.

The previous theorem implies this is independent of the choice ~~of~~ $G^\bullet$.

Ex: Take $D_{R/K}$ as a free cyclic module. Ber is good filtration w.r.t. Ber.

$$gr^{Ber}(D_{R/K}) \simeq K[y, z]$$ 2n variables std graded.

$$d(D_{R/K}) = 2n$$
$$e(D_{R/K}) = n.$$

Ex: Take $M = R$. $M \simeq D / D\langle\{\frac{\partial}{\partial x_i}\}\rangle$.
cyclic generated by 1.

$G^i := B^i \cdot 1$ is a good filtration

$$\| $$

$$\bigoplus_{|a|+|b| \leq i} \left(K \cdot \underline{x}^a \frac{\partial^{b_1}}{\partial x_1} \cdots \frac{\partial^{b_n}}{\partial x_n}\right)(1)$$

$$= \bigoplus_{|a| \leq i} (K \cdot \underline{x}^a)(1) = \bigoplus_{|a| \leq i} K \cdot x^\alpha = [R]_{\leq i}.$$

Then $d(R) =$ "usual dimension of R" $= n$

$e(R) =$ "usual multiplicity of R" $= 1$.

Exercise: For $M = D / D \cdot \langle x_1, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \rangle$,

what ~~is~~ are $d(M)$, $e(M)$?

Can you recognize M?