Prop: A ring, $R = A[\underline{x}]$ poly ring, $f \in R$ nonzerodivisor, $D = D_{R|A}$.

Then the map

$$((f) :_D (f)) \xrightarrow{\alpha} ((f) :_D (f))^{op}$$

given by $\alpha(\delta) = \bar{f}\, \Psi(\delta)\, \bar{f}^{-1}$ is an iso, where

$$D \xrightarrow{\Psi} D^{op} \text{ constructed last time.}$$

Pf: Can check that Ψ extends to an iso. $D_{R_f|A} \to D_{R_f|A}^{op}$ by $\Psi(\bar{r}\,\partial^{(\alpha)}) = (-1)^{|\alpha|}\partial^{(\alpha)}\,\bar{r}$ for $r \in R_f$, any α.

$$\delta \in ((f) :_D (f)) \iff \delta \bar{f}(R) \subseteq fR$$

$$\implies \bar{f}^{-1}\delta\bar{f}(R) \subseteq R$$

So $\bar{f}^{-1}\delta\bar{f} \in D$.

Then $\overset{\overset{D}{\downarrow}}{\gamma}(\bar{f}^{-1}\delta\bar{f}) = \gamma(\bar{f}^{-1}) * \gamma(\delta) * \gamma(\bar{f})$

$$= \gamma(\bar{f})\gamma(\delta)\gamma(\bar{f}^{-1})$$

$$= \bar{f}\gamma(\delta)\bar{f}^{-1} = \alpha(\delta).$$

so $\alpha(\delta) \in D$.

Then $\alpha(\delta)\cdot(fR) = \bar{f}\gamma(\delta)\bar{f}^{-1}(fR)$

$$\subseteq \bar{f}\gamma(\delta)(R) \subseteq \bar{f}R,$$

so α is well-defined.

Easy to see α is additive.

$$\alpha(\delta\varepsilon) = \bar{f}\,\psi(\delta\varepsilon)\,\bar{f}^{-1} = \bar{f}\,\psi(\varepsilon)\psi(\delta)\,\bar{f}^{-1}$$

$$= \underbrace{\bar{f}\,\psi(\varepsilon)\,\bar{f}^{-1}}_{\alpha(\varepsilon)}\,\underbrace{\bar{f}\,\psi(\delta)\,\bar{f}^{-1}}_{\alpha(\delta)}$$

$$= \alpha(\delta) * \alpha(\varepsilon).$$

So α is a homomorphism.

Then $\alpha^2(\delta) = \alpha(\bar{f}\,\psi(\delta)\,\bar{f}^{-1})$

$$= \bar{f}\,\psi(\bar{f}\,\psi(\delta)\bar{f}^{-1})\,\bar{f}^{-1}$$

$$= \bar{f}\,\psi(\bar{f}^{-1})\;\psi\psi(\delta)\;\psi(\bar{f})\,\bar{f}^{-1}$$

$$= \bar{f}\,\bar{f}^{-1}\;\psi^2(\delta)\;\bar{f}\,\bar{f}^{-1}$$

$$= \delta.$$

Thus, α is an isomorphism. ∎

Note: Symmetry properties of differential operator rings holds more generally, e.g., for R fin. gen. graded k-algebra that is Gorenstein, one has $D_{R|k} \cong D_{R|k}^{op}$. (Quinlan-Gallego).

We conclude:

Thm (Tripp): Let k be a field of char 0, $R = \frac{k[x,y]}{(xy)}$. Then $((xy) :_{D_{k[x,y]|k}} (xy))$

is left and right Noetherian, and hence, so is $D_{R^G|K}$. $\blacksquare$

Let • K be a field of characteristic 0,

• R poly ring over K

• G finite group acting linearly on R with no pseudoreflections.

<u>Thm</u> [Wallach]: In this setting, $D_{R^G|K}$ is D-algebra simple.

Pf: Let $J \leq D_R G$ be a nonzero two-sided ideal. Let $\delta \in J \setminus 0$ be of minimal order. Then, for $f \in R^G$,

$$[\delta, f] = \delta f - f \delta \in J$$

and has lower order, so must be zero; thus $\delta = r \in J$ for some $r \in R^G$.

We showed that $D_{R|k}$ is a f.g. right $D_{R^G|k}$-module

[Using that the same was true for $gr^{ord}(D_{R^G|k}) \subseteq gr^{ord}(D_{R|k})$
is
$gr^{ord}(D_{R|k})^G$
by Kantor's theorem]

Write $D_{R/k} = \sum_i \gamma_i D_{RG/k}$ for $\gamma_1, \dots, \gamma_t \in D_{R/k}$

and $N = \max\{\operatorname{ord}(\gamma_i)\} + 1$.

Set $\gamma_i^{(0)} := \gamma_i$, $\gamma_i^{(j)} := [\gamma_i^{(j-1)}, \bar{r}]$

inductively, so, in particular,

$\gamma_i^{(N)} = 0$ for i.

<u>Claim</u>: For each k and any $\delta \in D_{R/k}$, there are $c_1, \dots, c_k \in \mathbb{Z}$ with

$$\bar{r}^k \gamma_i = \gamma_i \bar{r}^k + c_1 \gamma_i^{(1)} \bar{r}^{k-1} + \dots + c_k \gamma_i^{(k)}.$$

<u>prf of claim</u>: By induction on k with $k=0$ trivial.

Note that $\bar{r}\gamma_i^{(j)} = \gamma_i^{(j)}\bar{r} - \gamma_i^{(j+1)}$,

so, for inductive step,

$$\bar{r}^{k+1}\gamma_i = \bar{r}\gamma_i\bar{r}^k + c_1\bar{r}\gamma_i^{(1)}\bar{r}^{k-1} + \cdots + c_k\bar{r}\gamma_i^{(k)}$$

$$= (\gamma_i\bar{r} - \gamma_i^{(1)})\bar{r}^k + c_1(\gamma_i^{(1)}\bar{r} - \gamma_i^{(2)})\bar{r}^{k-1} + \cdots$$

$$= \gamma_i\bar{r}^{k+1} + (c_1 - 1)\gamma_i^{(1)}\bar{r}^k + \cdots$$

$$+ (c_k - c_{k-1})\gamma_i^{(k)}\bar{r}$$

$$- c_k\gamma^{(k+1)}.$$

✓claim.

Using observation & claim, we have $\bar{r}^N\gamma_i \in D_{R|K}\cdot\bar{r}$ (left ideal) for each i.

Now, R is D algebra simple, i.e., $D_{R|K}$ is simple.

Thus, $1 \in \underbrace{D_{R|K}\cdot\bar{r}^N\cdot D_{R|K}}_{\text{two-sided ideal gen by } \bar{r}^N \neq 0}$

$$\subseteq D_{R|K} \cdot \overline{\Gamma}^{U}\left(\sum \gamma_i D_{R^G|K}\right)$$

$$\subseteq D_{R|K} \cdot \overline{\Gamma} \cdot D_{R^G|K}.$$

That is, $1 = \sum_i \alpha_i \overline{\Gamma} \beta_i$ $\quad \alpha_i \in D_{R|K}$

$\beta_i \in D_{R^G|K}$

$\shortparallel$

$(D_{R|K})^G.$

Now, consider the map

$$\rho : D_{R|K} \longrightarrow D_{R|K}^G = D_{R^G|K}$$

given by $\rho(\delta) = \frac{1}{|G|} \sum_{g \in G} g \cdot \delta.$

Note that $\rho(1) = 1.$

Further, this is a right $D_{R^G|K}$-mod homomorphism:

if $\quad \delta \in D_{R|K}, \; \varepsilon \in D_{R|K}^{G}$, then

$$\rho(\delta\varepsilon) = \frac{1}{|G|}\sum_{g\in G} g\cdot(\delta\varepsilon)$$

$$= \frac{1}{|G|}\sum_{g\in G}(g\cdot\delta)(g\cdot\varepsilon)$$

$$= \frac{1}{|G|}\sum_{g\in G}(g\cdot\delta)\,\varepsilon$$

$$= \rho(\delta)\cdot\varepsilon.$$

$\bar{r} \in D_{R^G}, \; \beta_i \in D_{R^G}$

Thus, $\quad 1 = \rho(1) = \rho\Big(\sum_i \underbrace{\alpha_i}_{\delta}\, \underbrace{\bar{r}\beta_i}_{\varepsilon}\Big)$

$$= \sum_i \rho(\alpha_i \bar{r}\beta_i) = \sum_i \underbrace{\rho(\alpha_i)}_{\in D_{R^G}}\,\bar{r}\underbrace{\beta_i}_{\in D_{R^G}}$$

$$\in J.$$

Thus $J = D_{R^G|K}$. $\blacksquare$

Cor: If R, K, G as above, then any nonzero local cohomology module on R^G is faithful. ∎

Exercise: Let $R = \mathbb{C}[x^2, xy, y^2]$.

Find explicit operators in $D_{R|\mathbb{C}}$ that show $1 \in D_{R|\mathbb{C}} \cdot \bar{x}^2 \cdot D_{R|\mathbb{C}}$.

I.e., find $\alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_t \in D_{R|\mathbb{C}}$ such that $1 = \sum_{i=1}^{t} \alpha_i \cdot \bar{x}^2 \cdot \beta_i$.

Good filtrations

Def: Let $(T, F^\bullet)$ be a filtered ring, and M a left (right) T-module. A filtration $G^\bullet$ on M is a <u>good filtration</u> if $gr(M, G^\bullet)$ is a f.g. $gr(T, F^\bullet)$-mod.

prop: Let $(T, F^\bullet)$ be a filtered ring, with $gr(T, F^\bullet)$ fin. gen. commutative K-algebra. Then

M is a fin gen left (right) T-mod

$\Updownarrow$

M admits a good filtration.

Pf: ($\Uparrow$) If M has a good filtration, then $gr(M, G^\bullet)$ f.g. $gr(T, F^\bullet)$-module. Then, a lift of the generators of $gr(M, G^\bullet)$ to M forms a

$m + G^{i-1} \rightsquigarrow m$
$m \in G^i$

generating set for M as a T-module.

($\Downarrow$) Given $\{m_1, \dots, m_t\}$ gen set for M, set $G^i := \sum_j F^i \cdot m_j$.

This is clearly ascending, satisfies

$$F^a \cdot G^b = \sum_j F^a F^b \cdot m_j \subseteq \sum_j F^{a+b} m_j = G^{a+b},$$

and $\bigcup_i G^i = M$ since $\{m_1, \dots, m_t\}$ generate.

Show that $gr(M, G^\bullet)$ is finitely generated over $gr(T, F^\bullet)$

(exercise.)

<u>Prop:</u> Let $(T, F^\bullet)$ be a filtered k-alg with $gr(T, F^\bullet)$ f.g. commutative k-alg.

Let M be a left (right) T-mod.

Let $G^\bullet$ be a good filtration on M, $H^\bullet$ any filtration on M.

Then $\exists a \in \mathbb{N}$ s.t. $G^i \subseteq H^{i+a}$ for all i.

<u>Pf:</u> Pick $m_1, \dots, m_t \in M$ s.t.

$\overline{m_1} = m_1 + G_{a_1 - 1}, \dots, \overline{m_t} = m_t + G_{a_t - 1}$ generate $gr(M, G^\bullet)$ as a $gr(T, F^\bullet)$-module.

Let $b_1, \dots, b_\ell$ be s.t. $m_i \in H^{b_i} \setminus H^{b_i - 1}$ for each i. The assumption on generation implies that

$$G_t = \sum_i F_{t-a_i} \cdot m_i \text{ for each } t.$$

Then, for $t > \max\{a_i\}$,

$$G_t = \sum F_{t-a_i} m_i \subseteq \sum F_{t-a_i} H^{b_i}$$
$$\subseteq \sum H^{t+b_i-a_i} \subseteq H^{t+a}$$

for $a = \max\{b_i - a_i\}$. $\boxtimes$

<u>Prop</u>: Let $(T, F^\bullet)$ be a filtered k-algebra with $\mathrm{gr}(T, F^\bullet)$ f.g. commut. k-alg.

M left (right) T-mod with $G^\bullet, H^\bullet$ good filtrations. Then $\exists\, c$ s.t.

$$G^{i-c} \subseteq H^i \subseteq G^{i+c}$$ for all i. $\boxed{}$

[Follows from last proposition by switching roles & taking maximum.]

Morally, all good filtrations are "same up to a shift".

For f.g. modules, get notion of filtration that is unique/

well-defined enough to preserve certain properties/invariants.