Last time: Defined opposite rings,
which are "same" except with multiplication
$r \ast s = sr$.

and $(x):(x)$ right Noetherian

$\boxed{(xy):(xy)}$ right Noetherian

$\Downarrow$

$\boxed{D_{\frac{k[x,y]}{(xy)}|k}}$ right Noetherian

Goal: to show $((xy):(xy)) \simeq ((xy):(xy))^{op}$.

Want to see this symmetry property for polynomial rings first.

Rmk: If $\alpha: T \longrightarrow T^{op}$ homomorphism, then $\alpha: T^{op} \longrightarrow T$ is also a homom.
clear ~~for~~ $+, -$, and
$\alpha(r \ast s) = \alpha(sr) = \alpha(s) \ast \alpha(r)$
$= \alpha(r)\alpha(s)$

A ring isomorphism $T \to T^{op}$ is also called an <u>antiisomorphism</u> $T \to T$.

<u>Lem</u>: Let $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{N}^n$.

1) $\binom{\alpha+\beta}{\gamma} = \sum_{\delta+\varepsilon=\gamma} \binom{\alpha}{\delta}\binom{\beta}{\varepsilon}$

2) $\sum_{\beta \le \alpha} (-1)^{\beta} \binom{\alpha}{\beta} = 0$ if $\alpha \neq 0$

($\beta_i \le \alpha_i$ each i). ($= 1$ if $\alpha = 0$)

<u>pf</u>: Exercise (follow from usual binomial coeff. identities).

<u>Prop</u>: Let A be a commut. ring, $R = A[\underline{x}]$ poly,

then $\partial^{(\alpha)} f = \sum_{\beta+\gamma=\alpha} \partial^{(\beta)}(f) \cdot \partial^{(\gamma)}$ in $D_{R|A}$.

<u>Prf</u>: First, let $f = x^{\mu}$ monomial.

suffices to check the equality by plugging in x^{σ} to both sides.

We have:

$$(\partial^{(\alpha)}\overline{x}^{\mu})(x^{\sigma}) = \binom{\mu+\sigma}{\alpha} x^{\mu+\sigma-\alpha}$$

and

$$\sum_{\beta+\gamma=\alpha} \overline{\partial^{(\beta)}(x^{\mu})}\, \partial^{(\gamma)}(x^{\sigma})$$

$$= \sum_{\beta+\gamma=\alpha} \overline{\binom{\mu}{\beta} x^{\mu-\beta}} \binom{\sigma}{\gamma} x^{\sigma-\gamma}$$

$$= \sum_{\beta+\gamma=\alpha} \binom{\mu}{\beta}\binom{\sigma}{\gamma} x^{\mu+\sigma-\alpha}$$

equal by lemma.

For general $f = \sum_{\sigma} a_{\sigma} x^{\sigma}$,

$$\partial^{(\alpha)}\overline{f} = \partial^{(\alpha)}\left(\overline{\sum_{\sigma} a_{\sigma} x^{\sigma}}\right)$$

$$= \sum_{\sigma} a_{\sigma}\, \partial^{(\alpha)}\overline{x}^{\sigma} = \sum_{\sigma} a_{\sigma}\left(\sum_{\beta+\gamma=\alpha} \overline{\partial^{(\beta)}(x^{\sigma})}\, \partial^{(\gamma)}\right)$$

$$= \sum_{\beta+\gamma=\alpha} \overline{\partial^{(\beta)}(f)}\, \partial^{(\gamma)}$$

∎

Prop (Quinlan-Gallego): For $R = A[x]$ poly ring, the map

$$D_{R|A} \xrightarrow{\Psi} D_{R|A}^{op} \text{ given by}$$

$$\Psi(\bar{r}\,\partial^{(\alpha)}) = (-1)^{|\alpha|}\,\partial^{(\alpha)}\,\bar{r}$$

is a ring isomorphism.
(where $|\alpha| = \alpha_1 + \cdots + \alpha_n$).

Does this specify a unique map?
Yes, since $D_{R|A} = \bigoplus_\alpha \bar{R}\,\partial^{(\alpha)}$!

Pf: Need to show this is multiplicative. Since any elt. is a sum of elements of the form $\bar{r}\,\partial^{(\alpha)}$, suffices to show

$$\Psi(\bar{r}\,\partial^{(\alpha)}\bar{s}\,\partial^{(\beta)}) \overset{?}{=} \Psi(\bar{r}\,\partial^{(\alpha)}) * \Psi(\bar{s}\,\partial^{(\beta)})$$

$$\overset{?}{=} \Psi(\bar{r}) * \Psi(\partial^{(\alpha)}) * \Psi(\bar{s}) * \Psi(\partial^{(\beta)})$$

so, suffices to show

i) $\Psi(\bar{r}\delta) = \Psi(\bar{r}) \circledast \Psi(\delta)$ any $r \in R$, $\delta \in D_{R|A}$

ii) $\Psi(\delta \partial^{(\alpha)}) = \Psi(\delta) \circledast \Psi(\partial^{(\alpha)})$ any α, $\delta \in D_{R|A}$

iii) $\Psi(\partial^{(\alpha)} \bar{r}) = \Psi(\partial^{(\alpha)}) \circledast \Psi(\bar{r})$ any α, $r \in R$.

i) We can write, WLOG, $\delta = \bar{s}\partial^{(\alpha)}$.

$$\Psi(\bar{r}\,\bar{s}\,\partial^{(\alpha)}) = (-1)^{|\alpha|}\partial^{(\alpha)}\,\overline{rs} = \left((-1)^{|\alpha|}\partial^{(\alpha)}\bar{s}\right)\bar{r}$$
$$= \Psi(\bar{r}) \circledast \Psi(\bar{s}\partial^{(\alpha)})$$

ii) Similar.

iii) $$\Psi(\partial^{(\alpha)}\bar{r}) = \Psi\Big(\sum_{\beta+\gamma=\alpha} \overline{\partial^{(\beta)}(r)}\,\partial^{(\gamma)}\Big)$$

$$= \sum_{\beta+\gamma=\alpha} (-1)^{|\gamma|}\,\partial^{(\gamma)}\,\overline{\partial^{(\beta)}(r)}$$

$$= \sum_{\beta+\gamma=\alpha} (-1)^{|\gamma|}\Big(\sum_{\delta+\varepsilon=\gamma} \overline{\partial^{(\delta)}(\partial^{(\beta)}(r))}\,\partial^{(\varepsilon)}\Big)$$

$$= \sum_{\beta+\delta+\varepsilon=\alpha} (-1)^{|\delta+\varepsilon|}\,\overline{\binom{\beta+\delta}{\beta}\partial^{(\beta+\delta)}(r)}\,\partial^{(\varepsilon)}$$

$$= (-1)^{|\alpha|} \sum_{\varepsilon+\xi=\alpha} \Bigg(\underbrace{\sum_{\beta+\delta=\xi} (-1)^{|\beta|} \binom{\xi}{\beta}}_{} \Bigg) \overline{\partial^{(\xi)}(r)}\, \partial^{(\varepsilon)}$$

By Lemma $= 0$ for $\xi \neq 0$
and $= 1$ for $\xi = 0$.

so this is just $(-1)^{|\alpha|}\, \overline{r}\, \partial^{(\alpha)}$.

Thus, Ψ is multiplicative, so is a homomorphism.

Then $\Psi : D_{R|A}^{op} \longrightarrow D_{R|A}$ is also a homomorphism.

$$\Psi^2(\overline{r}) = \overline{r}$$

$$\Psi^2(\partial^{(\alpha)}) = \Psi\big((-1)^{|\alpha|}\partial^{(\alpha)}\big)$$

$$= (-1)^{|\alpha|}(-1)^{|\alpha|}\partial^{(\alpha)} = \partial^{(\alpha)}.$$

Thus, $\Psi^2 = \mathrm{id.}$ on $D_{R|A}$, so

Ψ is an isomorphism. $\boxtimes$

<u>Conclusion</u>: Flipping mult. order and switching sign on $\partial/\partial x_i$ is antiisomorphism of $D_{R|A}$.

This is sometimes called the Fourier transform on $D_{\mathbb{C}[\underline{x}]|\mathbb{C}}$.

Now, show some symmetry property for the operators that preserve a prinicpal ideal.

<u>Prop</u>: Let A be a ring, $R = A[\underline{x}]$ poly ring, and $f \in R$ nonzerodivisor. Then the map $(\langle f \rangle :_{D_{R|A}} \langle f \rangle) \xrightarrow{\alpha} (\langle f \rangle :_{D_{R|A}} \langle f \rangle)^{op}$

given by $\alpha(\delta) = f\, \Psi(\delta)\, f^{-1}$

is an isomorphism, where

$\Psi : D_{R|A} \to D_{R|A}^{op}$ is the previous iso.

prf: Given $\delta \in (f):(f)$, we have $\delta(\overline{f}R) \subseteq fR$, so

$$(\delta \overline{f})(R) \subseteq fR.$$

Note that since $f \in R$ is a nonzero divisor, the map $\boxed{fR \xrightarrow{\overline{f}^{-1}} R}$ is well-defined.

Then

$$(\overline{f}\,\psi(\delta)\,\overline{f}^{-1})\cdot(fR)$$

$$= (\overline{f}\,\psi(\delta))(R) \subseteq fR,$$

so the map is well-defined.

What is

$$fR \xrightarrow{\overline{f}^{-1}} R \ ?$$

Given $fr \in fR$, set

$$\overline{f}^{-1}(fr) = r.$$

Is well-defined, since

$$fr = fs \Rightarrow f(r-s) = 0 \Rightarrow r-s=0 \Rightarrow r=s.$$

Is also R-linear, since

$$\begin{tikzcd} \mathfrak{f}sr \arrow[r, mapsto] & sr \\ \mathfrak{f}r \arrow[r, mapsto] \arrow[u, "\cdot s"] & r \arrow[u, "\cdot s"'] \end{tikzcd}$$

Can also think of this as taking place inside of $R_{\mathfrak{f}}$: have

$$\begin{tikzcd} R \arrow[r, phantom, "\subseteq"] & R_{\mathfrak{f}} \\ \mathfrak{f}R \arrow[u, dashed] \arrow[r, phantom, "\subseteq"] & R_{\mathfrak{f}} \arrow[u, "\cdot \mathfrak{f}^{-1}"'] \end{tikzcd}$$

$$(\mathfrak{f}\,\Psi(\delta)\,\mathfrak{f}^{-1})(R) \subseteq R.$$

Have $\mathfrak{f}^{-1} \in D_{R_{\mathfrak{f}}|A}$; check containment above in $D_{R_{\mathfrak{f}}|A}$.

$$(\mathfrak{f}\,\underbrace{\Psi(\delta)})\underbrace{\boxed{\left(\frac{r}{\mathfrak{f}}\right)}} \in R.$$

$$\left(\frac{\partial}{\partial x_1}\right)\left(\frac{r}{f}\right) = \frac{\frac{\partial r}{\partial x_1} f - \frac{\partial f}{\partial x_1} r}{f^2}$$

might not be in $\frac{1}{f}R$.

Come back to this later.

$R = k[x]$ Q: Is $\alpha(\underline{\delta})(\underline{R}) \leq \underline{R}$?

$f = x$

$\alpha(\delta) = \bar{x}\, \psi(\delta)\, \bar{x}^{-1}$

$\delta := r\,\partial^{(\beta)}$

$\alpha(r\,\partial^{(\beta)}) = \bar{x}\, (-1)^{|\beta|} \cdot \bar{r}\, \bar{x}^{-1}$

Q: Is $\alpha(\partial^{(\beta)}) \in D_{R|K}$? ✓ special case

$$\alpha(\partial^{(\beta)}) = \left(\overline{x}\, \partial^{(\beta)}\, \overline{x}^{-1} \right)$$

$$\alpha(\partial^{(\beta)}) \cdot 1 = \cancel{\overline{x}}\, \partial^{(\beta)}(1/x)$$

$$= \overline{x}\, \frac{1}{\beta!}\left(\frac{\partial}{\partial x}\right)^{\beta}(1/x)$$

$$= \overline{x} \cdot \frac{1}{\beta!} \frac{(-1 \cdot -2 \cdots -\beta)}{x^{\beta+1}}$$

$$= x^{-\beta}.$$

Only need, for $\delta \in (f) :_{D_{R_f|A}} (f)$, that $\delta(R) \subseteq R$ ...