In Lucas' theorem, we had

$$\boxed{\prod_{i=0}^{k} \sum_{n_i=0}^{p-1} \binom{m_i}{n_i} X^{n_i p^i}} = \sum_{n=0}^{m} \left(\prod_{i=0}^{k} \binom{m_i}{n_i} \right) X^n$$

$m = \sum_{i=0}^{k} m_i p^i$ fixed $(0 \le m_i < p)$

$$\left\{ \begin{array}{c} (n_0, \dots, n_k) \\ 0 \le n_i < p \end{array} \right\} \xleftrightarrow{\text{bij}} \{ 0 \le n < p^{k+1} \}$$

$$(n_0, \dots, n_k) \longmapsto \sum_{i=0}^{k} n_i p^i$$

$$\sum_{\substack{(n_0, \dots, n_k) \\ 0 \le n_i < p}} \binom{m_0}{n_0} \cdots \binom{m_k}{n_k} X^{n_0 p^0} \cdots X^{n_k p^k}$$

$$= \sum_{(n_0, \dots, n_k)} \left(\prod_{i=0}^{k} \binom{m_i}{n_i} \right) X^{\left(\sum_{i=0}^{k} n_i p^i \right)}$$

$$= \sum_{n=0}^{p^{k+1}-1} \left(\prod_{i=0}^{k} \binom{m_i}{n_i} \right) X^n$$

If $n > m$, then $n_i > m_i$ for some i, else if $m_i \le n_i$ all i, then $m = \sum_i m_i p^i \le \sum_i n_i p^i = n$.

So if $n > m$, $\prod_{i=0}^{k} \binom{m_i}{n_i}$ is a product where at least one term is ~~zero~~, so it is zero.

Last time, we saw ~~that~~ for $R = \frac{K[x,y]}{(xy)}$, K field of char 0, $gr^{ord}(D_{R|K})$ is not Noetherian.

Today, with same R, K, will see $D_{R|K}$ is left Noeth. and right Noeth.

From earlier, we have

$$((x) :_{D_{K[x]|K}} (x)) = K \oplus \bigoplus_{\substack{i>0 \\ j\geq 0}} x^{-i}\left(\frac{\partial}{\partial x}\right)^j$$

$$((xy) :_{D_{K[x,y]|K}} (xy)) = K \oplus \bigoplus_{\substack{i>0 \\ j\geq 0}} x^{-i}\frac{\partial^j}{\partial x^j} \oplus \bigoplus_{\substack{i>0 \\ j\geq 0}} y^{-i}\frac{\partial^j}{\partial y^j} \oplus \bigoplus_{\substack{i>0 \\ j>0 \\ a,b}} x^{-i}y^{-j}\frac{\partial^a}{\partial x^a}\frac{\partial^b}{\partial y^b}$$

$$\simeq ((x) :_{D_{K[x]|K}} (x)) \otimes_K ((y) :_{D_{K[y]|K}} (y)).$$

As $D_{R|K}$ is a quotient ring of $(xy):(xy)$, it suffices to show that $(xy):(xy)$ is left/right Noetherian.

<u>Lem</u>: $((x):(x))$ is right Noetherian.

<u>Pf</u>: Call $A := (x):(x)$, which is a subring of $D := D_{k[x]/k}$.

Note that $D = A \oplus \bigoplus_{i>0} k \cdot \left(\frac{\partial}{\partial x}\right)^i$, as k-vectorspaces. Let $J \subseteq A$ be a right ideal; want to see ~~that~~ J is fin. gen.

Since D is right Noetherian, there are finitely many elements $\underline{f} = f_1, \dots, f_n \in J$ s.t. $(\underline{f})D = JD$.

If $(\underline{f})A = J$, we are done.

If $(\underline{f})A \neq J$, pick $\beta \in J \setminus (\underline{f})A$.

Since $\beta \in J \subseteq JD = (\underline{f})D$, can write

$$\underset{\cap\, J}{\beta} = \underbrace{\alpha}_{\in J} + \underbrace{\gamma_1 \frac{\partial}{\partial x} + \cdots + \gamma_r \left(\frac{\partial}{\partial x}\right)^r}_{\in J}$$

with $\alpha \in \underset{\cap\, J}{(\underline{f})} A$, $\gamma_i \in \underset{\cap\, J}{(\underline{f})} \cdot k$

Claim: $\gamma_i \frac{\partial}{\partial x} \in \underbrace{(\underline{f}) \frac{\partial}{\partial x} \cdot k}_{} \cap J$. $\xleftarrow{\text{no } i \text{ here}}$

for each i.

prf of claim: Just need to see that each is in J. Will do a trick.

$$\sum_{i=1}^{r} i \gamma_i \left(\frac{\partial}{\partial x}\right)^{i-1} = \sum_{i=1}^{r} \gamma_i \left(\left(\frac{\partial}{\partial x}\right)^i x - x\left(\frac{\partial}{\partial x}\right)^i\right)$$

$$\underset{\cap\, J}{\gamma_1} + \sum_{i=2}^{r} i \gamma_i \left(\frac{\partial}{\partial x}\right)^{i-1} = \underbrace{-\sum_{i=1}^{r} \underset{\cap\, J}{\gamma_i} \underbrace{x \left(\frac{\partial}{\partial x}\right)^i}_{\in A}}_{\in J} + \underbrace{\underbrace{\left(\sum_{i=1}^{r} \gamma_i \left(\frac{\partial}{\partial x}\right)^i\right)}_{\in J} \underset{\cap\, A}{x}}_{\in J}$$

$$\Rightarrow \sum_{i=2}^{r} i \gamma_i \left(\frac{\partial}{\partial x}\right)^{i-1} \in J$$

Repeat: $$\sum_{i=2}^{r} i(i-1)\gamma_i \left(\frac{\partial}{\partial x}\right)^{i-2} = \sum_{i=2}^{r} i\gamma_i \left(\underbrace{\left(\frac{\partial}{\partial x}\right)^{i-1}\bar{x}}_{J} - \underbrace{\bar{x}\left(\frac{\partial}{\partial x}\right)^{i-1}}_{\in J}\right)$$

(last sum times $\bar{x}$)

$$\Longrightarrow \sum_{i=3}^{r} i(i-1)\gamma_i \left(\frac{\partial}{\partial x}\right)^{i-2} \in J$$

$$\vdots$$

$$r!\,\gamma_r\left(\frac{\partial}{\partial x}\right) \in J$$

$$\Longrightarrow \gamma_r\left(\frac{\partial}{\partial x}\right) \in J.$$

Then $$\gamma_r\left(\frac{\partial}{\partial x}\right)^r = \gamma_r\left(\frac{\partial}{\partial x}\bar{x} - \bar{x}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)^r$$

$$= \underbrace{-\gamma_r}_{\in J}\underbrace{\bar{x}\left(\frac{\partial}{\partial x}\right)^{r+1}}_{\in A} + \underbrace{\gamma_r\frac{\partial}{\partial x}}_{\in J}\underbrace{\bar{x}\left(\frac{\partial}{\partial x}\right)^{r}}_{\in A}$$

$$\in J.$$

Thus, $$\beta - \gamma_r\left(\frac{\partial}{\partial x}\right)^r = \alpha + \gamma_1\frac{\partial}{\partial x} + \cdots + \gamma_{r-1}\left(\frac{\partial}{\partial x}\right)^{r-1}$$
$$\in J.$$

By same argument (decreasing induction on ~~the~~ i), get that each $\gamma_i \frac{\partial}{\partial x} \in J$.

$\boxed{\text{claim} \checkmark}$

Then, the claim implies that J is generated by $(\underline{f})$ and the fin. dim. vectorspace

$$J \cap (\underline{f})\frac{\partial}{\partial x} \cdot K$$

Put together, get finite generating set for J. ☒.

If T is a noncommutative ring, then we can a poly ring over T with commuting variables $T[\underline{x}]$.

If T is an algebra over a field K, then $T[\underline{x}] \simeq T \otimes_K K[\underline{x}]$.

<u>Exercise</u>: If T is left/right Noetherian, then $T[\underline{x}]$ is left/right Noetherian.

[<u>Hint</u>: Usual proof of Hilbert Basis theorem.]

<u>Thm</u> [Tripp]: $((xy):(xy))$ is right Noetherian. Hence, $D_{\frac{K[x,y]}{(xy)}/K}$ is right Noetherian.

<u>prf</u> (sketch): Let $S = ((y) :_{D_{K[y]/K}} (y))$.

Then $((xy):(xy)) \simeq S \otimes_K ((x):(x))$.

Call this ring A. Note that A is a subring of $D = S \otimes_k D_{k[x]/k}$.

Proceed similarly to the previous lemma...

* Need to see that D is right Noetherian: filter D by $F^i = S \otimes_k D^i_{k[x]/k}$.

$\rightsquigarrow \operatorname{gr} F^i \simeq S \otimes_k \operatorname{gr} D^{\bullet}_{k[x]/k}$

$\simeq S[z_1, z_2]$ poly ring over S.

$\rightsquigarrow$ right Noeth by exercise

$\rightsquigarrow$ D is right Noeth.

Then, some computational trick shows that

$$J = (\underline{f})A + \underbrace{\left((\underline{f})\tfrac{\partial}{\partial x}(S \otimes 1) \cap J\right)}_{\text{submodule of f.g. right module} \Rightarrow \text{f.g.}} A$$

$\Rightarrow J$ is f.g. $\blacksquare$

We also want to see left Noetherian.

Will use opposite rings to see this.

Def: The opposite ring of a noncommutative ring T is the ring T^{op}, which as additive groups is identical to T, and has multiplication "$\star$"

$$r \star s = sr$$

T-multiplication.

will use the convention that $\star$ means "op" multiplication and usual multiplication notation means usual T-multiplication.

There is a natural bijection between left T-modules and right T^{op}-modules:

if M is a left T-module, then it is a right T^{op}-module by

$$m \,\square\, t := t \cdot m$$

left T-action on M.

$m \in M \quad t \in T^{op} \, (\underset{\text{sets}}{=} T)$

since

$$m \,\square\, (t \star s) = m \,\square\, (st)$$

$$= (st) \cdot m = s \cdot (t \cdot m)$$

$$= (m \,\square\, t) \,\square\, s \qquad \checkmark$$

In particular,

left ideals of T $\longleftrightarrow$ right ideals of T^{op}

so T is left Noeth
$\iff T^{op}$ is right Noeth.

Note also, $(T^{op})^{op} = T$ as rings.