Last time:

- If $(M, G^\bullet)$ is a filtered left/right $(T, F^\bullet)$-module, then a generating set for $\mathrm{gr}(M, G^\bullet) \rightsquigarrow$ generating set for M.
- If $\mathrm{gr}(T, F^\bullet)$ is left/right Noetherian, then T left/right Noetherian.
- R poly ring field k of char 0
  $\Rightarrow \mathrm{gr}^{\mathrm{ord}}(D_{R|k})$ Noeth
  $\Rightarrow D_{R|k}$ left & right Noeth.

related result for invariant rings of ~~finite~~ groups in char 0.

Now: Poly rings in char $p>0$.

Lem: (Lucas' theorem): Let p be prime, and

$$m = \sum_{i=0}^{k} m_i p^i$$

$$n = \sum_{i=0}^{k} n_i p^i \qquad 0 \le m_i, n_i < p$$

(base p expansions).

Then $\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \mod p$.

(We take $\binom{m}{n} = 0$ for $m < n$).

<u>prf:</u> In $\mathbb{F}_p[x]$, We have

$$\sum_{n=0}^{m} \binom{m}{n} x^n = (1+x)^m = (1+x)^{(\sum m_i p^i)}$$

$$= \prod_{i=0}^{k} \left((1+x)^{p^i}\right)^{m_i} = \prod_{i=0}^{k} (1+x^{p^i})^{m_i}$$

$$= \prod_{i=0}^{k} \sum_{n_i=0}^{m_i} \binom{m_i}{n_i} x^{n_i p^i} = \prod_{i=0}^{k} \sum_{n_i=0}^{p-1} \binom{m_i}{n_i} x^{n_i p^i}$$

$$= \sum_{n=0}^{m} \left(\prod_{i=0}^{k} \binom{m_i}{n_i} \right) x^n$$

each n has unique base p expansion.

$\boxtimes$

Cor: If $\exists e$: $a, b < p^e$, $a+b \geq p^e$, then $\binom{a+b}{a} \equiv 0 \mod p$.

prf: We claim that there is a base p digit of $a+b$ that is smaller than the corresponding digit of a.

Otherwise, setting $c = a+b$,

$$c = c_e p^e + c_{e-1} p^{e-1} + \cdots + c_0$$
$$a = \qquad\qquad a_{e-1} p^{e-1} + \cdots + a_0$$

$c_e > 0$, $0 \leq a_i \leq c_i < p$ for $i < e$.

$$b = b_e p^e + b_{e-1} p^{e-1} + \cdots + b_0$$

$b_i = c_i - a_i$ for all i is base p expansion.

This is contradicts that $b < p^e$.
Done by Lucas' Theorem. $\boxtimes$

<u>Notation:</u> For n-tuples α, β,

we will write $\alpha! = \alpha_1! \cdots \alpha_n!$

and $\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n} = \dfrac{\beta!}{\alpha!\,(\beta-\alpha)!}$.

<u>Lem:</u> Let $R = A[\underline{x}]$ poly ring.

Then $\partial^{(\alpha)} \partial^{(\beta)} = \binom{\alpha+\beta}{\alpha} \partial^{(\alpha+\beta)}$.

Pf: Evaluate at x^γ:

$$\partial^{(\alpha)} \partial^{(\beta)} (x^\gamma) = \partial^{(\alpha)} \left(\binom{\gamma}{\beta} x^{\gamma-\beta} \right)$$

$$= \binom{\gamma-\beta}{\alpha} \binom{\gamma}{\beta} \; x^{\gamma-(\alpha+\beta)}$$

$$= \frac{\cancel{(\gamma-\beta)!}\; \gamma!}{(\gamma-(\alpha+\beta))!\,\alpha!\,\cancel{(\gamma-\beta)!}\,\beta!} \; x^{\gamma-(\alpha+\beta)}$$

$$\partial^{(\alpha+\beta)} (x^\gamma) = \binom{\gamma}{\alpha+\beta} x^{\gamma-(\alpha+\beta)}$$

$$= \frac{\gamma!}{(\gamma-(\alpha+\beta))!\,(\alpha+\beta)!} \; x^{\gamma-(\alpha+\beta)}.$$

$$\binom{\alpha+\beta}{\alpha}\,\partial^{(\alpha+\beta)}(x^{\gamma}) = \frac{\cancel{(\alpha+\beta)!}\;\gamma!}{\alpha!\,\beta!\,(\gamma-(\alpha+\beta))!\,\cancel{(\alpha+\beta)!}}\,x^{\gamma-(\alpha+\beta)}$$

Since the operators agree on A-module gen set for R and they are A-linear, they agree as functions (operators) on R. $\boxed{}$

Th'm: Let K be a field of char $p>0$, and $R=K[\underline{x}]$ poly ring. Then $D_{R|K}$ is not left Noetherian.

Pf: Let $J_e = D_{R|K} \cdot \left\{ \partial^{(\alpha)} \;\middle|\; \begin{matrix} \alpha_1 > 0 \\ \alpha_i < p^e \\ \text{for each } i \end{matrix} \right\}$.

These form an ascending chain of left ideals.

We have $\partial^{(p^e,0,\dots,0)} \notin J_a$ $a<e$.

Write $D_{R|K} = \bigoplus R\,\partial^{(\beta)}$

Then $D_{R|k} \cdot \left\{ \partial^{(\alpha)} \;\middle|\; \begin{matrix} \alpha_1 > 0 \\ \alpha_i < p^e \end{matrix} \right\}$

$$= \sum \bar{R}\, \partial^{(\beta)} \partial^{(\alpha)}$$

$$= \sum \binom{\alpha+\beta}{\alpha} \bar{R}\, \partial^{(\alpha+\beta)}$$

For $\alpha+\beta = (p^e, 0, \ldots, 0)$, have $\binom{\alpha+\beta}{\alpha} = 0$ in R

$\Rightarrow$ ~~proper~~ ascending infinite chain of left ideals. ☒

$\dfrac{\mathbb{C}[x,y]}{(xy)}$ ← will come back to this one.

Ex: Let $R = \dfrac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$

$D_{R|\mathbb{C}}$ is ~~not~~ left Noetherian;

even worse, there is an infinite ascending proper chain of two-sided ideals!

$$J_k = \mathbb{C}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right][D^k]_1 + [D]_{\geq 2}$$

pos deg.

is a two-sided ideal since $[D]_{<0} = 0$, $[D]_0 = \mathbb{C}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right]$.

J_k a proper chain by earlier claim.

<u>Next goal:</u> For k field of char 0, $R = k[x,y]/(xy)$,

- $\mathrm{gr}^{\mathrm{ord}}(D_{R|k})$ is <u>not</u> Noetherian
- $D_{R|k}$ is left & right Noeth.

Def: Given two left/right $(T, F^\bullet)$-modules, $(M, G^\bullet)$, $(N, H^\bullet)$, we say a T-module homomorphism $M \xrightarrow{\varphi} N$ is a <u>map of filtered modules</u> or a <u>$(T, F^\bullet)$-mod homom.</u> if

$$\varphi(G^i) \subseteq H^i \text{ for all } i.$$

Exercise: If $(L, E^\bullet)$, $(M, G^\bullet)$, $(N, H^\bullet)$ are $(T, F^\bullet)$-modules (left or right), and

$$0 \to L \to M \to N \to 0$$

is a SES, where each map is a map of $(T, F^\bullet)$-modules, then

$$0 \to gr(L, E^\bullet) \to gr(M, G^\bullet) \to gr(N, H^\bullet) \to 0$$

is a SES of $gr(T, F^\bullet)$-modules.

Recall that if $S = A[x]$ poly ring, and $R = S/I$ then

$$D^i_{R|A} \simeq \frac{(I :_{D^i_{S|A}} I)}{I D^i_{S|A}} \simeq \frac{(I :_{D^i_{S|A}} I)}{(I :_{D^i_{S|A}} S)}$$

Then

$$D_{R|A} \simeq \frac{(I :_{D_{S|A}} I)}{(I :_{D_{S|A}} S)}$$

$(I :_{D_{S|A}} I)$ is a subring of $D_{S|A}$ ✓

and $(I :_{D_{S|A}} S)$ is a two-sided ideal of $(I :_{D_{S|A}} I)$:

$$(I : I)(I : S) \subseteq (I : S)$$

$$(I : S)\underset{\substack{\cup \\ (I:I)}}{(S : S)} \subseteq (I : S)$$

This preserves the order filtration, so by exercise, get SES

$$0 \to I\,\mathrm{gr}^{ord}(D_{S|A}) \longrightarrow \mathrm{gr}^{ord}(I :_{D_{S|A}} I) \to \mathrm{gr}^{ord}(D_{R|A}) \to 0$$

and $\mathrm{gr}^{ord}(I :_{D_{S|A}} I) \hookrightarrow \mathrm{gr}^{ord}(D_{S|A})$.

Back to $R = \dfrac{K[x,y]}{(xy)}$.

We computed earlier that

$$((xy) :_{D_{K[x,y]|K}} (xy)) = K \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} x^i \frac{\partial^j}{\partial x} \oplus \bigoplus_{\substack{i\geq 0\\ j\geq 0}} y^i \frac{\partial^j}{\partial y} \oplus \underbrace{\bigoplus_{i,j>0} x^i y^j \frac{\partial^a}{\partial x}\frac{\partial^b}{\partial y}}_{I D_{S|K}}.$$

$\mathrm{gr}^{ord}((xy):(xy)) \hookrightarrow \mathrm{gr}^{ord}(D_{K[x,y]|K})$ is $K[x,y,u,v]$, $v = \left[\frac{\partial}{\partial y}\right] + D^0$,

$x = [\bar{x}]$, $y = [\bar{y}]$, $u = \left[\frac{\partial}{\partial x}\right] + D^0$.

Have $\operatorname{gr}^{\mathrm{ord}}((xy):(xy))$

$$\simeq K \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} x^i u^j \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} y^i v^j \oplus \bigoplus_{\substack{i>0\\ j>0}} x^i y^j u^a v^b$$

$$\subseteq K[x,y,u,v]$$

$$\rightsquigarrow \operatorname{gr}^{\mathrm{ord}}(D_{R|K}) \simeq \frac{\operatorname{gr}^{\mathrm{ord}}((xy):(xy))}{(xy)}$$

$$\simeq K \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} x^i u^j \oplus \bigoplus_{\substack{i>0\\ j\geq 0}} y^i v^j \subseteq \frac{K[x,y,u,v]}{(xy)}$$

$$\simeq K\left[\begin{matrix} x, xu, xu^2, xu^3, \dots \\ y, yv, yv^2, yv^3, \dots \end{matrix}\right] \subseteq \frac{K[x,y,u,v]}{(xy)}.$$

(nonNoetherian commutative ring.

quotient ring

$$K[x, xu, xu^2, xu^3, \dots] \subseteq K[x,u].$$

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