$J \ni \delta \neq 0$

$\boxed{\delta \in \mathrm{Hom}_{R^{p^e}}(R,R)}$

$R \simeq \bigoplus R^{p^e} x^{\alpha}$

Want to show

$1 \in D_{R|k} \cdot \delta \cdot D_{R|k} (\subseteq J)$

r^{p^e}

δ "is" a matrix $p^e n \times p^e n$

with entries in R^{p^e}

r^{p^e} some nonzero entry.

$$\begin{array}{ccc} R^{p^e} & \simeq & R \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ r^{p^e} & \longmapsto & r \end{array}$$

r is part of a free basis for R as an R^{p^e}-module

$$\begin{array}{ccc} R & \xrightarrow{\;r\;} & R \\ \text{is} & \xleftrightarrow{\text{can choose these}} & \text{is} \\ (R^{p^e})^{\oplus m} & & (R^{p^e})^{\oplus m} \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} R^{p^e} & \xrightarrow{\;r^{p^e}\;} & R^{p^e} \\ \text{is} & & \text{is} \\ (R^{p^{e+e}})^{\oplus m} & & (R^{p^{e+e}})^{\oplus m} \end{array}$$

so that r is part of the free basis.

here too

$\Longrightarrow$ 1 is an entry in the matrix.

Think of

$$\begin{array}{ccc} R & \xrightarrow{\;s\;} & R \\ \text{is} & & \text{is} \\ (R^{p^{e+e}})^{\oplus m} & & (R^{p^{e+e}})^{\oplus m} \end{array}$$

"is a direct summand of"

$$\begin{tikzcd}
R \arrow[r, "\text{matrix}"] \arrow[d, phantom, "\cong" rotate=90] & R \arrow[d, phantom, "\cong" rotate=90] \\
\left((R^{p^e})^{\oplus n}\right) \arrow[r, "\text{block decomposition}"] \arrow[d, phantom, "\cong" rotate=90] & (R^{p^e})^{\oplus n} \arrow[d, phantom, "\cong" rotate=90] \\
\left((R^{p^{e+c}})^{\oplus M}\right) \arrow[r] & R^{p^{e+c}}
\end{tikzcd}$$

← direct sum decomposition of R.

one of these blocks has a 1.

$A \to R$ D-algebra simple

M any D-module

$\Longrightarrow \operatorname{ann}_{D_{R|A}}(M)$ is a two-sided ideal,

so either (0) or $D_{R|A}$.

If $r \overset{\neq 0}{\in} \operatorname{ann}_R(M)$, then for all $m \in M$

$$0 = \underset{\substack{\uparrow \\ R\text{-action}}}{r \cdot m} = \underset{\substack{\uparrow \\ D_{R/A}\text{-action}}}{\bar{r} \cdot m},$$

~~so~~ $\bar{r} \in \operatorname{ann}_{D_{R/A}}(M)$, ~~then~~

(D-alg simplicity) $D_{R/A} = \operatorname{ann}_{D_{R/A}}(M)$

So $M = 0$.

$R = K[x, xy, y^2, y^3]_{(x, xy, y^2, y^3)}$ ← an

is ~~not~~ CM but

is CM on spec°

Is a 2-dim domain.

$\mathfrak{p} \subsetneq \mathfrak{m} \Rightarrow R_{\mathfrak{p}}$ is a domain of dim ≤ 1.

Filtrations & Noetherianity

<u>Recall</u>: $(T, F^\bullet)$ is a <u>filtered ring</u> if T is a ring with $F^0 \subseteq F^1 \subseteq F^2 \subseteq \cdots$ abelian groups s.t.

- $\bigcup_i F^i = T$ (exhaustive)
- $F^i F^j \subseteq F^{i+j}$ (multiplicative).

If T is an A-algebra, $(T, F^\bullet)$ a <u>filtered A-algebra</u> if also

- $A \subseteq F^0$

($\Rightarrow$ each F^i is an A-module).

If M is a left ~~(right)~~ T-module and $(T, F^\bullet)$ is filtered ring, then $G^\bullet$ is a filtration on M consistent with $F^\bullet$ or $(M, G^\bullet)$ is a (filtered) left (right) $(T, F^\bullet)$-module if

- $F^i G^j \subseteq G^{i+j}$ (right modules $\rightsquigarrow$ $G^j F^i \subseteq G^{i+j}$).

If $(T, F^\bullet)$ is a filtered A-algebra then $gr(T, F^\bullet) = \bigoplus_i F^i / F^{i-1}$ is a graded A-algebra with $A \subseteq gr(T, F^\bullet)_0$.

If $(M, G^\bullet)$ is a $\overset{\text{left(right)}}{}$ $(T, F^\bullet)$-module, then $gr(M, G^\bullet) = \bigoplus_i G^i / G^{i-1}$ is a graded left (right) $gr(T, F^\bullet)$-module.

<u>Ex</u>: k field of char 0

$R = k[\underline{x}]$ poly ring.

Then $(D_{R|k}, D^\bullet_{R|k})$ is a filtered
(order filtration)
k-algebra, and

$$gr^{ord}(D_{R|k}) := gr(D_{R|k}, D^\bullet_{R|k})$$

$$\cong k[y_1, \dots, y_n, z_1, \dots, z_n] \quad \text{(poly ring.)}$$

with $y_i = \overline{x_i}$ degree 0

$z_i = \overline{\frac{\partial}{\partial x_i} + D^0_{R|k}}$ degree 1.

More generally, write
$gr^{ord}(D_{R|A})$ for $gr(D_{R|A}, D^\bullet_{R|A})$.

Recall that $\mathrm{gr}^{\mathrm{ord}}(D_{R|A})$ is always commutative.

<u>Lem</u>: Let k be a field of char 0, and G a finite group. Then the functor

$$\underset{\text{(invariants)}}{(-)^G} : \underset{\left(\substack{\text{representations of}\\ G \text{ over } k}\right)}{k[G]\text{-mod}} \longrightarrow \underset{(k\text{-vector spaces})}{k\text{-mod}}$$

is exact.

<u>pf</u>: In general (no assumption on characteristic), the invariants functor is left-exact [exercise].

To see exactness, need to check it preserves surjections.

~~Given~~ a $k[G]$-module M, there is

a projection map $p_M : M \longrightarrow M^G$

$$p_M(m) = \frac{1}{|G|}\sum_{g\in G} g\cdot m.$$

For a $K[G]$-linear map $M \xrightarrow{\alpha} N$, have commutative diagram

$$\begin{tikzcd} M \arrow[r, two heads, "\alpha"] \arrow[d, "p_M"] & N \arrow[d, "p_N"] \\ M^G \arrow[r, "\alpha|_{M^G}"] & N^G \end{tikzcd}$$

Thus, if α is surjective, $\alpha|_{M^G}$ is surjective as well. $\square$

Ex: Let K be a field of char 0. Let $R = K[\underline{x}]$ and

prior lemma

G be a finite group acting linearly on R with no pseudoreflections. kantor

Then G acts on $D_{R|K}$ by conjugation and preserves the order filtration:

$$g(D^i_{R|K}) \subseteq D^i_{R|K} \text{ for each } i.$$

We have $\underline{D^i_{R^G|k} \simeq (D^i_{R|k})^G}$ by Kantor's theorem. Since G preserves order filtration, G acts on

$gr^{ord}(D_{R|k})$ by $g \cdot (\delta + D^{i-1}_{R|k})$
$$= (g \cdot \delta) + D^{i-1}_{R|k}.$$

So, $0 \to D^{i-1}_{R|k} \to D^i_{R|k} \to gr^{ord}(D_{R|k})_i \to 0$

as $k[G]$-modules.

Then,
$$gr^{ord}(D_{R^G|k})_i \simeq \frac{D^i_{R^G|k}}{D^{i-1}_{R^G|k}} \simeq \frac{(D^i_{R|k})^G}{(D^{i-1}_{R|k})^G} \simeq \left(\frac{D^i_{R|k}}{D^{i-1}_{R|k}}\right)^G$$
$$\simeq gr^{ord}(D_{R|k})^G_i.$$

So $gr^{ord}(D_{R^G|k}) \simeq \underbrace{\boxed{gr^{ord}(D_{R|k})}^G}_{\text{poly ring!}}.$

By Noether's finiteness theorem for invariants (on polynomial rings), $\mathrm{gr}^{\mathrm{ord}}(D_{R^G|k})$ is a fin. gen. k-algebra, hence Noeth., and $\mathrm{gr}^{\mathrm{ord}}(D_{R^G|k}) \hookrightarrow \mathrm{gr}^{\mathrm{ord}}(D_{R|k})$ is mod-finite.

Exercise: Let $R = \mathbb{C}[\overset{y}{x^2}] \subseteq S = \mathbb{C}[x]$.

Then $R = S^G$ where $G = \{1, g\}$ with $g \cdot x = -x$, and the operator $\frac{\partial}{\partial y} \in D_{R|k}$ does not extend to a differential operator on S.

($\Longrightarrow$ "no pseudoreflections" is necessary in Kantor's theorem).

Prop: Let $(T, F^\bullet)$ be a filtered ring and $(M, G^\bullet)$ be a left/right $(T, F^\bullet)$-module. Let $m_1, \ldots, m_t \in M$ be such that

$$m_1 + G^{d_1 - 1}, \ldots, m_t + G^{d_t - 1} \in gr(M, G^\bullet)$$

form a generating set as a left/right $gr(T, F^\bullet)$-module. Then $m_1, \ldots, m_t$ form a gen. set for M as a left/right T-module.

prf: By hypothesis, we have

$$gr(M, G^\bullet)_n = \sum_i gr(T, F^\bullet)_{n-d_i} (m_i + G^{d_i - 1}), \text{ so}$$

$$G^n / G^{n-1} = \sum_i F^{n-d_i} / F^{n-d_i-1} \cdot (m_i + G^{d_i - 1}), \text{ so}$$

$G_n = \left(\sum_i F_{n-d_i} \cdot m_i \right) + G_{n-1}$, for each n.

Thus, $G_n \subseteq \sum_i T m_i + G_{n-1}$ for each n.

Then for $n=0$, $G_{-1}=0$, so

$G_0 \subseteq \sum_i T m_i$, and if $G_{n-1} \subseteq \sum_i T m_i$

then $G_n \subseteq \sum_i T m_i$, so by

induction on n, the m_i's generate $\blacksquare$

A ring T is <u>left Noetherian</u>

if the following equivalent conditions hold:

i) any ascending chain of left ideals stabilizes,

ii) every nonempty family of left ideals has a max'l elt.

iii) every left ideal is f.g.

iv) every left submodule of a f.g. left module is f.g.

v) every f.g. left module is fin. pres.

Pf: Exercise (similar to commutative case).

Prop: If $(T, F^\bullet)$ is a filtered ring and $gr(T, F^\bullet)$ is left (right) Noeth then T is left (right) Noeth.

Pf: If $J \subseteq T$ is a left ideal, then $(J, J \cap F^\bullet)$ is a filtered left $(T, F^\bullet)$-module.

In this case

$$gr(J, J\cap F^\bullet) \hookrightarrow gr(T, F^\bullet),$$

so this identifies with a left ideal, which by hypothesis is f.g.

Then by prev. prop. J is f.g., so T is left Noeth. $\boxtimes$

<u>Thm</u>: Let k be a field of char 0, R poly ring over k, then $D_{R|k}$ is left and right Noeth. $\boxtimes$

<u>Thm</u>: ~~With~~ k, R as above, if G is finite, acts linearly on R with no pseudoreflections, then $D_{R^G|k}$ is left and right Noeth., and $D_{R|k}$ is a fin. pres. left/right $D_{R^G|k}$-module. $\boxtimes$

Rmk: It is ~~not~~ true that

$D_{R|k}$ (left) Noeth $\Rightarrow gr^{ord}(D_{R|k})$ Noeth.

For example, for $R = \mathbb{C}[x,y]/(xy)$

i) $gr^{ord}(D_{R|\mathbb{C}})$ is not Noeth.

ii) $D_{R|\mathbb{C}}$ is both left- and right-Noetherian.

For all SR rings R

$D_{R|k}$ is right-Noeth,

but some, not all, are left-Noeth.

e.g. $R = \dfrac{\mathbb{C}[x,y,u,v]}{(xu, xv, yu, yv)}$ has $D_{R|\mathbb{C}}$ not left-Noeth.

[Tripp].