Now: want to show that polynomial rings are D-algebra simple.

will have different proofs in char 0 and char $p>0$.

Recall/exercise: For a poly. ring over a field,

$$[\underline{x}^\alpha \partial^{(\beta)}, x_i] = \underline{x}^\alpha \partial^{(\beta - e_i)}$$

$$[\underline{x}^\alpha \partial^{(\beta)}, \partial^{(e_i)}] = -\alpha_i \underline{x}^{\alpha - e_i} \partial^{(\beta)}$$

where $e_i = (0, \dots, 0, \underset{\uparrow\, i\text{th spot}}{1}, 0, \dots, 0)$

Thm: K field of char 0, $R = K[\underline{x}]$ poly ring. Then R is D-algebra simple.

Pf: Let $J \neq 0$ be a two-sided ideal of $D_{R|K}$, and $\delta \in J$ nonzero.

For $\gamma \in D_{R|K}$, note that $[\delta, \gamma] \in J$.

Write $\delta = \sum_i \lambda_i \underline{x}^{\alpha_i} \partial^{(\beta_i)}$ for some $\lambda_i \in K^\times$.

Reorder so that $|\beta_1| \geq |\beta_i|$ for each i.
Apply $[-, \bar{x}_j]$ β_{1j} times for each j.
We then get $\bar{r} \in J \setminus \{0\}$. Then apply $[-, \partial^{(e_j)}]$ repeatedly to get some $\bar{\lambda} \in J \setminus \{0\}$, so $\bar{I} \in J$. $\boxed{}$.

Matrix rings: If R is a commutative ring, and F is a free module of rank n, then a choice of basis for F (i.e. an iso $F \simeq R^{\oplus n}$) induces an isomorphism $\mathrm{End}_R(F) \simeq \mathrm{Mat}_{n \times n}(R)$
$(= \mathrm{Hom}_R(F, F))$

"Left multiplication" in $\mathrm{Mat}_{n\times n}(R)$
$\Updownarrow$
row operations

"right multiplication" in $\mathrm{Mat}_{n\times n}(R)$
$\Updownarrow$
column operations.

Given a matrix M with a nonzero entry r in any position, can generate (as a two-sided ideal) all matrices with entries in (r). Likewise, if the entries of M generate $I \subseteq R$, then M generates (as a two-sided ideal)

$\to$ $\mathrm{Mat}_{n\times n}(I) \subseteq \mathrm{Mat}_{n\times n}(R)$. All two-sided ideals arise this way.

In particular: 1) $\mathrm{Mat}_{n\times n}(R)$ is generally not simple (unless R is a field).

2) A ~~gener~~ matrix $M \in \mathrm{Mat}_{n\times n}(R)$ generates the whole ~~matrix~~ ring as a two-sided ideal if it has a unit entry.

An element $\varphi \in \mathrm{End}_R(F)$ (F free module) generates the whole endo. ring if it has a unit entry with respect to any free basis for F.

Thm. Let k ~~be~~ a perfect field of char $p > 0$, and $R = k[x]$ poly ring. Then R is D-algebra simple.

Pf: Let $J \neq 0$ be a two-sided ideal, $\delta \in J \setminus \{0\}$.
We have $\delta \in \mathrm{Hom}_{R^{p^e}}(R, R)$ for some e (and all larger e).
R free R^{p^a}-mod of finite rank
$\Rightarrow \mathrm{Hom}_{R^{p^a}}(R, R) \cong \mathrm{Mat}_{?\times ?}(R^{p^a})$.
Thus, if δ considered as a matrix in $\mathrm{Hom}_{R^{p^a}}(R, R)$ (for some a, and some choice of free basis) has a unit entry, then $I \in D^{(a)} \cdot \delta \cdot D^{(a)}$
$\subseteq D_{R|K} \cdot \delta \cdot D_{R|K} \subseteq J$.
First, consider $\delta \in \mathrm{Hom}_{R^{p^e}}(R, R)$ as a matrix with entries in R^{p^e}, and let r^{p^e} be an entry. Note that r is part of a free basis for R as a free R^{p^c}-module for some c.

Thus $\boxed{\bar{r} \in \mathrm{Hom}_{R^{p^c}}(R,R)}$ has 1 as ~~a~~ entry in its matrix for some basis.

We saw this on Monday: given a poly $r \in R$, can choose e large enough so that r becomes ~~part of~~ a free basis

Likewise, $\bar{r}^{p^e} \in \mathrm{Hom}_{R^{p^{e+c}}}(R^{p^e}, R^{p^e})$ has 1 as an entry in some basis $\{f_\alpha\}$ for R^{p^e} over $R^{p^{e+c}}$.

$$\begin{array}{ccccc} R & \geq & R^{p^e} & \geq & R^{p^{e+c}} \\ & \{g_\beta\} & & \{f_\alpha\} & \end{array}$$

Now if $\{g_\beta\}$ is a free basis for R over R^{p^e},

then $\{f_\alpha g_\beta\}$ is a free basis for R over $R^{p^{e+c}}$.

Then in this basis, the matrix for δ has 1 as an entry. $\boxed{\checkmark}$

Cor: Let R be a polynomial ring over a perfect field k. Then every local cohomology module $H^i_I(R)$ is either zero or faithful (as an R-module). $\boxtimes$

Rmk/Exercise: The perfect field hypothesis can be removed, e.g., by a faithfully flat base change argument.

Now, want to show that

D-algebra simple $\Rightarrow$ Cohen Macaulay.

Def: A local ring $(R, \mathfrak{m})$ is Cohen-Macaulay (CM) if $\operatorname{depth}_{\mathfrak{m}}(R) = \dim(R)$.

A ring R is CM if $R_{\mathfrak{p}}$ is CM for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Facts: 1) The ring definition does not contradict the local definition:

$$\operatorname{depth}_{\mathfrak{m}}(R) = \dim(R) \implies \operatorname{depth}_{\mathfrak{p}}(R_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}})$$

$(R, \mathfrak{m})$ local $\qquad$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

2) $(R, \mathfrak{m})$ is CM $\iff H^{<\dim(R)}_{\mathfrak{m}}(R) = 0$

$\iff H^{<ht(\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$ all primes $\mathfrak{p}$.

3) If $(R, \mathfrak{m})$ is local, ess. of finite type over a field k and $R_{\mathfrak{p}}$ is CM for all $\mathfrak{p} \neq \mathfrak{m}$, then $H^i_{\mathfrak{m}}(R)$ has finite length as an R-module for $i < \dim(R)$.

Thm (van den Bergh): Let R be ess of fin type over a field k and suppose that R is D-algebra simple. Then R is CM.

Pf: If not, pick $\mathfrak{p} \in \mathrm{Spec}(R)$ with $R_\mathfrak{p}$ not CM, but $R_\mathfrak{q}$ CM for all $\mathfrak{q} \subsetneq \mathfrak{p}$. [We can do this since R is Noetherian, so $\mathrm{Spec}(R)$ satisfies the descending chain condition. Thus, if $\{\mathfrak{p} \mid R_\mathfrak{p} \text{ not CM}\}$ is nonempty, it has a minimal element.]

Then $\exists i < \mathrm{ht}(\mathfrak{p})$ with $H^i_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) \neq 0$ and has finite length as an $R_\mathfrak{p}$-module.

Thus, $\mathfrak{p}^n \cdot H^i_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) = 0$.

Note that $H^i_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) = H^i_\mathfrak{p}(R_\mathfrak{p})$ (since they are computed by the exact same Čech complex).

Then $H^i_\mathfrak{p}(R_\mathfrak{p})$ is a nonzero $D_{R|K}$-module that is not R-module faithful. Thus R is not D-algebra simple. $\boxtimes$

Recall:

$$\begin{tikzcd}
D\text{-algebra simple} \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow] & D\text{-module simple} \arrow[l, Rightarrow, shift left, "/" description] \\
CM & \text{direct summands of polynomial rings} \arrow[u, Rightarrow] \\
 & \left(\text{classical invariant rings}\right) \arrow[u, Rightarrow]
\end{tikzcd}$$

<u>Conjecture</u> (Levasseur–Stafford):

Classical invariant rings (in characteristic zero) are D-algebra simple.

Many cases are known (LS, Schwarz), but this is an open question still.

We will do finite group invariants later.

The characteristic p analogue of LS conj. was settled by Smith–VandenBergh.