Def: • R is F-finite if

$F: R \to R$ is mod-fin

$\Longleftrightarrow$ F is alg-fin

$\Longleftrightarrow$ F^e is mod-fin for all $\Longleftrightarrow$ some e

$\Longleftrightarrow$ 〃 alg-fin

• R is F-split if

if $F: R \to R$ splits as a map of R-mods

($\Longleftrightarrow$ $F^e: R \to {}^eR$ splits for all $\Longleftrightarrow$ some e)

$\Longrightarrow$ R is reduced

• R is <u>strongly F-regular</u> if $\forall c$ not in any minimal prime of R, $\exists e$ s.t.

$$R \xrightarrow{cF^e} {}^eR$$
$$r \longmapsto cr^{p^e}$$ splits as R-modules.

($\Rightarrow$ R is F-split)

<u>Facts:</u> 1) regular + F-finite $\Rightarrow$ str F-reg

2) str F-reg $\Leftrightarrow$ product of str F-reg domains

<u>Ex:</u> Let k be a perfect field of char $p>0$. Let $R = k[\underline{x}]$ be a poly ring. Then

$$\boxed{R = \bigoplus_{\substack{0 \le \alpha_i < p^e \\ \text{each } i}} x^\alpha R^{p^e}} \quad \Leftrightarrow \quad {}^eR = \bigoplus_{\substack{0 \le \alpha_i < p^e \\ \text{each } i}} x^\alpha R$$

$R^{p^e} = k[x_1^{p^e}, \dots, x_n^{p^e}]$

$(R^{p^e})^\times = k^\times$

$\Rightarrow$ R is F-finite.

every nonzero monomial in c has every exponent less than p^e.

If $c \in R \setminus \{0\}$, then $\exists e$ s.t. $\cancel{c \notin (x_1^{p^e}, \ldots, x_n^{p^e})}$.

This means that ~~some~~ every coefficient of c in the basis $\{\underline{x}^\alpha \mid \text{each } i \ 0 \le \alpha_i < p^e\}$ is a unit in R^{p^e}, say $\underline{x}^\beta$. Then c is part of a free basis for R as an R^{p^e}-module, e.g., $\{c\} \cup \{\underline{x}^\alpha \mid \alpha \neq \beta\}$ is a free basis. Then projection ~~onto~~ that coordinate is an R^{p^e}-linear map

$$\begin{aligned} R &\longrightarrow R^{p^e} \\ c &\longmapsto 1 \end{aligned}$$

This map is a splitting of cF^e, so R is strongly F-regular.

every nonzero monomial in c has every exponent less than p^e. $\rightsquigarrow$ in the basis $\{x^\alpha \mid 0 \le \alpha_i < p^e \text{ each } i\}$

$$c = \sum_\alpha \underset{\in R^{p^e}}{\gamma_\alpha} x^\alpha \qquad \gamma \in K.$$

these are the coefficients.

Thm (Smith): Let R be F-split and ess. of finite type over a perfect field. Then R is str. F-reg $\iff$ R is a product of D-simple domains.

Pf: By Fact Q' above, this reduces to the case of a domain. Then

$$R \text{ str. F-reg} \iff \forall c \neq 0, \exists e : \exists \varphi \in \mathrm{Hom}_{R^{p^e}}(R, R^{p^e}) \text{ s.t. } \varphi(c) = 1.$$

Then, postcomposing with inclusion $R^{p^e} \subseteq R$ yields $\tilde{\varphi} \in \mathrm{Hom}_{R^{p^e}}(R,R) = D^{(e)}$ with $\tilde{\varphi}(c)=1$, so R is D-module simple.

Conversely, if R is D-module simple, $\forall c \neq 0$ $\exists \tilde{\varphi} \in D^{(e)}$ with $\tilde{\varphi}(c) = 1$. Let $\beta : R \longrightarrow R^{p^e}$ R^{p^e}-linear with $\beta(1) = 1$.

Then $(\beta \circ \tilde{\varphi})(c) = 1$ and $\beta \circ \tilde{\varphi} \in \mathrm{Hom}_{R^{p^e}}(R, R^{p^e})$, so R is str. F-reg. $\square$

Cor: If R is ess. of finite type over k perfect of char $p>0$, and R strongly F-reg., then R is D-module simple.

Fact: In char $p>0$,

direct summand of regular ring $\Longrightarrow$ strongly F-regular
$\not\Longleftarrow$

D-algebra simplicity

Recall that a (noncommutative) algebra is simple if it admits no proper quotient; equivalently, it has no proper two-sided ideals.

If R is commutative,

simple $\Longleftrightarrow$ field.

Def: Given $A \to R$, we say R is D-algebra simple if $D_{R|A}$ is a simple A-algebra.

Prop: If R is D-algebra simple, then every nonzero D-module is a faithful R-module.

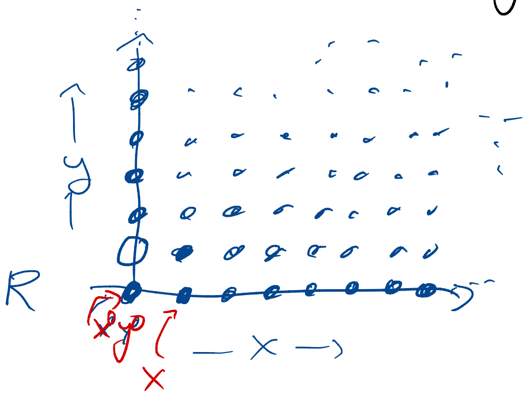
pf: We have $\mathrm{ann}_D(M)$ is a two-sided ideal. If $r \overset{\neq 0}{\in} \mathrm{ann}_R(M)$, then $\bar{r} \in \mathrm{ann}_D(M)$, so $D_{R|A} = \mathrm{ann}_D(M)$, so $M = 0$. ☒

Prop: If R is D-algebra simple, then R is D-module simple.

Pf: $I \subseteq R$ D-ideal $\Rightarrow$ R/I D-module with annihilator $I \Rightarrow I = (0)$ or R. ☒

Prop: If R is D-algebra simple, then for any ideal $I \subseteq R$ and any i, $H^i_I(R)$ is either a faithful R-module or zero. Likewise for $H^i_I(M)$ M any D-module. ☒

Ex: Let $R = \mathbb{C}[x, xy, y^2, y^3] \subseteq \mathbb{C}[x,y] = S$.



$$R = \bigoplus_{\substack{(i,j)\neq(0,1)\\ i,j\geq 0}} \mathbb{C}\cdot x^i y^j$$

$$\cap|$$

$$S = \bigoplus_{\substack{(i,j)\\ i,j\geq 0}} \mathbb{C}\cdot x^i y^j.$$

Claim: R is D-module simple but not D-algebra simple.

1) R is not D-algebra simple:

Consider the short-exact sequence of R-modules:

$$0 \to R \to S \to \mathbb{C}\cdot y \to 0$$

where $\mathfrak{m} = (x, xy, y^2, y^3)$ kills $\mathbb{C}\cdot y$.

$(\mathbb{C}\cdot y \simeq R/\mathfrak{m})$.

Note that $\sqrt{(x, y^2)} = \mathfrak{m}$.

Property of local cohomology: short-exact sequences of modules $\rightsquigarrow$ long exact sequences of cohomology

$$0 \to H^0_{(x,y^2)}(R) \to \underbrace{H^0_{(x,y^2)}(S)}_{0} \to H^0_{(x,y^2)}(\mathbb{C}\cdot y)$$

$$\to H^1_{(x,y^2)}(R) \to \underbrace{H^1_{(x,y^2)}(S)}_{0} \to H^1_{(x,y^2)}(\mathbb{C}\cdot y)$$

$$\to H^2_{(x,y^2)}(R) \to \cdots$$

Remark: $H^i_{(x,y^2)}(S) \simeq H^i_{(x,y)}(S)$ for each i.

Computation claimed earlier: $H^i_{(x,y)}(S) = 0$ for $i<2$.

[In general, $H^j_J(T) = 0$ for $j < \operatorname{depth}_J(T)$.]

$$H^0_{(x,y^2)}(\mathbb{C}\cdot y) = \left[\ker\left(\mathbb{C}\cdot y \to \underbrace{(\mathbb{C}\cdot y)_x}_{0} \oplus \underbrace{(\mathbb{C}\cdot y)_{y^2}}_{0}\right)\right]$$

$$\simeq \mathbb{C}\cdot y.$$

Get from LES:

$$0 \to \mathbb{C}\cdot y \to H^1_{(x,y^2)}(R) \to 0$$

so $H^1_{(x,y^2)}(R) \simeq \mathbb{C}\cdot y$ is nonzero, not faithful.

2) R is D-module simple.

Observe: $\operatorname{Frac}(R) \cong \operatorname{Frac}(S) = \mathbb{C}(x,y)$, and that

$$D_{R|\mathbb{C}} = \{\delta \in D_{\mathbb{C}(x,y)|\mathbb{C}} \mid \delta(R) \subseteq R\}$$
$$\cup\!|$$
$$\{\delta \in D_{S|\mathbb{C}} \mid \delta(R) \subseteq R\}.$$

Note that $\alpha = 1 - y\frac{\partial}{\partial y} \in D_{S|\mathbb{C}}$

can be computed as, for $f = \sum_i f_i(\cancel{x}) y^i \in S$,

$\alpha(f) = \sum_i (1-i) f_i(x) y^i \subseteq R$.

Also, $\alpha(1) = 1$.

Then, given $f \in R \setminus \{0\}$, $\exists\, \delta \in D_{S|\mathbb{C}}$ with $\delta(f) = 1$, since S is D-module simple.

Then, $(\alpha \circ \delta) \in D_{R|\mathbb{C}}$ (since $\alpha(S) \subseteq R$),

and $(\alpha \circ \delta)(f) = 1$, so R is D-module simple. $\square$