$$S = \mathbb{R}[\underline{x}]$$
$\underline{x}$: n of these

$$\operatorname{Hom}_{D_{S|\mathbb{R}}}\left(H^n_{(\underline{x})}(S), M\right) = 0.$$

for

$$M = \begin{cases} \mathbb{R}[\underline{x}] \\ \mathbb{R}[[\underline{x}]] \\ \mathbb{R}\{\underline{x}\} \\ C^\infty(\mathbb{R}^n) \end{cases}$$

because the only function (in M) with $x_i f = 0$ all x_i is zero function.

Equally well can look for solutions to differential equations in any D-module:

$$(\ast\ast) \quad \Big\{ \underbrace{\begin{bmatrix} \delta_{11} & \cdots & \delta_{1m} \\ \vdots & & \\ \delta_{n1} & \cdots & \delta_{nm} \end{bmatrix}}_{A} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

linear system of PDE's

$$\Updownarrow$$

$$N_A = \operatorname{coker}\left(D^n_{S|\mathbb{R}} \xrightarrow{A} D^m_{S|\mathbb{R}}\right)$$

fin pres. D-module

$$\rightsquigarrow$$

$$\mathrm{Sol}_{N_A}(M) := \mathrm{Hom}_{D_{S|\mathbb{R}}}(N_A, M)$$

$\mathrm{Sol}_{N_A}(-) : D\text{-mod} \longrightarrow \mathbb{R}\text{-vector space}$

can look for solutions to $\circledast$ in any D-module M.

D-ideals & D-simplicity

Recall: A D-ideal is a D-submodule of R.

Def: A comm. ring R is D-module simple (over $A \to R$) if it is simple as a D-module. I.e., the only D-ideals of R are (0) and R.

Remark: Many sources say "D-simple" for what we call "D-module simple."

Lem: R is D-mod-simple $\Longleftrightarrow$ $\forall r \in R \setminus \{0\}$ $\exists \delta \in D_{R|A} : \delta(r) = 1$.

pf: ($\Leftarrow$) If $I \neq (0)$ is a D-ideal, $r \in I \setminus \{0\}$, then $\exists\, \delta$: $\delta(r) = 1$. Then $1 \in I$, so $I = R$.

($\Rightarrow$) For $r \in R$, $D_{R|A}(r) \subseteq R$ is a D-ideal. If $r \neq 0$, then $D_{R|A}(r) = R$, so $\exists\, \delta$ s.t. $\delta(r) = 1$. $\blacksquare$

<u>Prop</u>: If R is Noeth., reduced, and D-mod. simple, (for any A), then R is a domain.

pf: In this case, (0) is radical, so it is the ~~intersection~~ intersection of all the minimal primes of R. Each of these is then a D-ideal, so there can only be just (0); i.e., (0) is prime. $\blacksquare$

<u>Recall</u>: In the Noeth. case, minimal primary of D-ideals are D-ideals:

$(I :_{D_{R|A}} I) \subseteq (Q :_{D_{R|A}} Q)$

if Q is a min primary cpt. of I.

Ex: ~~Let~~ R be a poly ring over a field k of char 0. Then R is D-mod. simple. In fact, we proved something much stronger:

$$D^i_{R|K} \xrightarrow{\text{res.}} \mathrm{Hom}_K([R]_{\leq i}, R)$$

is bijective. Thus, given $r \neq 0$, we have $r \in [R]_{\leq i}$ for some i, and there is some $\delta \in D^i_{R|K}$ with $\delta(r) = 1$.

In fact, the same argument works for any field of any characteristic.
(Exercise: double-check that that argument works char. free).

Ex: A Stanley-Reisner ring (other than a poly ring) is ~~not~~ D-module simple.

E.g., for $R = \dfrac{K[x,y]}{(xy)}$, $(0), (x), (y), (x,y)$ are all D-ideals.

Ex: The cubic $R = \mathbb{C}[x,y,z]/(x^3+y^3+z^3)$ is ~~not~~ D-mod simple. Indeed, there are no diff'l ops of negative degree, ~~so~~ each ideal $[R]_{\geq i}$ is a D-ideal.

Lem: Let $A \to R \to S$ comm. rings, M, N S-modules. Then

$$D^i_{S|R}(M,N) \subseteq D^i_{S|A}(M,N) \subseteq D^i_{R|A}(M,N)$$

pf: Exercise.

Lem: Let $A \longrightarrow R \xrightarrow{i} S$ comm. rings, and $\beta: S \longrightarrow R$ R-linear. Then there is a map

$M=N=S$ $\qquad$ $M=N=R$

$$\rho : D^i_{S|A} \longrightarrow D^i_{R|A}$$
$$\delta \longmapsto \beta \circ \delta \circ i.$$

(warning: ~~not~~ a ring homomorphism).

Pf: We have $D^i_{S|A} \subseteq D^i_{R|A}(S,S)$,
$\beta \in D^0_{R|A}(S,R)$, $i \in D^0_{R|A}(R,S)$,
~~so~~ follows from general composition rule. $\square$

Thm (Smith): Let $A \to R \to S$ comm. rings, S D-module simple (over A), and R a direct summand of S. Then R is D-module simple.

Pf: $R \xrightarrow{i} S$, $S \xrightarrow{\beta} R$ splitting
~~so~~ $\beta(1) = \beta(i(1)) = 1$. Then $\forall r \in R \setminus \{0\}$,
$i(r) \neq 0$ and $\exists \delta \in D_{S|A}$ with $\delta(i(r)) = 1$.
~~So~~, $\beta(\delta)(r) = \beta(\delta(i(r))) = \beta(1) = 1$. $\square$

Cor: A direct summand of a poly. ring over a field is D-mod. simple.

In particular, rings of invariants of linearly reductive groups,

$R \overset{\oplus}{\hookrightarrow} S$ direct summand

$\Rightarrow IS \cap R = I$ all ideals I.

I D_R-ideal

e.g. finite groups with $|G| \in k^\times$, $(k^\times)^t$, and others, are D-module simple.

IS $\mathcal{S}$? D_S-ideal dubious

...

~~Also~~ $R = \mathbb{C}[\{\Delta_{ij} \mid i<j\}]$ is D-mod simple.

Now, want to relate D-mod simplicity ~~to~~ to classes of singularities in positive characteristic.

Recall: $R \xrightarrow{F} R$

$F(r) = r^p$ is a ring homom. if char $p>0$.

Likewise $R \xrightarrow{F^e} R$, $F^e(r) = r^{p^e}$ its iterates.

We will write ${}^eR$ for R with the R-module structure via restr. of scalars through F^e. That is,

$$\underset{R}{s} \cdot \underset{{}^eR}{r} = \underset{{}^eR}{s^{p^e} r}$$

$R \curvearrowright {}^eR$

Thus, $R \xrightarrow{F^e} {}^eR$ is R-linear.

R is reduced $\iff$ F is injective

$\iff$ F^e is injective for some $\iff$ all e.

In this case (R reduced),

$\mathrm{Frac}(R) := W^{-1}R$, W set of nonzerodivisors on R

is a product of fields, and

$$R \hookrightarrow \mathrm{Frac}(R) \simeq \prod_{p \in \mathrm{Min}(R)} \mathrm{Frac}(R/p)$$

$$\downarrow$$

alg. closure

$$\overline{\mathrm{Frac}(R)} := \prod_{p \in \mathrm{Min}(R)} \overline{\mathrm{Frac}(R/p)}$$

Then, we have $R^{1/p^e} := \{ r \in \overline{\mathrm{Frac}(R)} \mid r^{p^e} \in R \}$.

is a subring ~~of~~ of $\overline{\mathrm{Frac}(R)}$, and the R-module structure on R^{1/p^e} via restriction of scalars $R \subseteq R^{1/p^e}$ agrees with R-module ${}^eR$ via $\underset{\in {}^eR}{r} \longmapsto \underset{\in R^{1/p^e}}{r^{1/p^e}} := \left(\begin{array}{l}\text{element } s \text{ such that} \\ s^{p^e} = r,\ s \in \overline{\mathrm{Frac}(R)}\end{array}\right)$.

Also in this case $R \simeq R^{p^e} \subseteq R$ and we can identify $R \xrightarrow{F^e} {}^eR$ with $R^{p^e} \xrightarrow{\text{inclusion}} R$.

$$\begin{tikzcd}
 & {}^eR \arrow[dr, "\simeq"] & \\
R \arrow[ur, "F^e"] \arrow[rr, "1"] \arrow[d, "F^e", "\wr"'] & & R^{1/p^e} \arrow[d, "F^e", "\wr"'] \\
R^{p^e} \arrow[rr, "1"] & & R
\end{tikzcd}$$

$$r \mapsto r^{p^e} \qquad r \mapsto r \qquad r \mapsto r^{p^e} \qquad r^{p^e} \mapsto r^{p^e}$$

Frobenius map $R \longrightarrow {}^eR$

$\wr$

inclusion $R \longrightarrow R^{1/p^e}$

$\wr$

inclusion $R^{p^e} \longrightarrow R$

$$\cdots \simeq R^{p^2} \underset{\text{rings}}{\overset{\xleftarrow{F}}{\simeq}} R^p \underset{\text{rings}}{\overset{\xleftarrow{F}}{\simeq}} R \underset{\text{rings}}{\overset{\xleftarrow{F}}{\simeq}} R^{1/p} \underset{\text{rings}}{\overset{\xleftarrow{F}}{\simeq}} R^{1/p^2} \underset{\text{rings}}{\simeq} \cdots$$