Last time:

M D-module, $W \subseteq R$ mult.closed

$\Rightarrow$ $W^{-1}M$ is a D-module in a unique way.

In particular, $W^{-1}R$ is always a D-module.

Prop: Let M be a D-module, and $\delta \in D^{i}_{R|A}$.
Then the map $M \xrightarrow{\delta \cdot} M$, $m \longmapsto \delta \cdot m$ is an element of $D^{i}_{R|A}(M, M)$.

Pf: By induction on i.

$i = 0 \Rightarrow \delta = \bar{r}$, which acts as $M \xrightarrow{\cdot r} M$ by definition, which is in $\operatorname{Hom}_R(M, M)$
$\qquad = D^{0}_{R|A}(M, M)$.

ind. step: $\delta \in D^{i}_{R|A}$. Want to see that
$[(m \mapsto \delta \cdot m), \bar{r}] \in D^{i-1}_{R|A}(M, M)$ for each $r \in R$.

This sends $m \longmapsto \delta \cdot (rm) - r(\delta \cdot m)$
$= [\delta, \bar{r}] \cdot m$,
which is in $D^{i-1}_{R|A}(M, M)$ by IH. $\boxtimes$

Note: if T is a noncomm. ring and M is a left (or right) T-module, then $\mathrm{ann}_T(M)$ is a two-sided ideal:

$$\alpha \cdot M = 0, \ \beta \in T \implies$$

$$(\alpha\beta)\cdot M = \alpha\cdot(\beta\cdot M) \subseteq \alpha \cdot M = 0$$

$$(\beta\alpha)\cdot M = \beta\cdot(\alpha\cdot M) = \beta\cdot 0 = 0.$$

Thus,

Prop: The annihilator of a D-module is a two-sided ideal of $D_{R|A}$. $\boxtimes$

Local cohomology

Given $\underline{f} = f_1, \dots, f_n$ sequence of elements and an R-module (R comm.) M, define $\check{C}^\bullet(\underline{f}; M)$ as the as the complex of R-modules

$$0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_n} \to 0$$

with $\pm$ signs chosen in such a way as to obtain a complex, e.g.,

$$0 \to M \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} M_{f_1} \oplus M_{f_2} \xrightarrow{\begin{bmatrix}1 & -1\end{bmatrix}} M_{f_1 f_2} \to 0$$

<u>Theorem:</u> If $\sqrt{(\underline{f})} = \sqrt{(\underline{g})}$, then

$$H^i(\check{C}^\bullet(\underline{f}; M)) \simeq H^i(\check{C}^\bullet(\underline{g}; M)).$$

$(\underline{f}) = (\underline{g}) \Rightarrow H^i(\check{C}^\bullet(\underline{f}; M)) \Leftarrow$ well-defined. is $H^i(\check{C}^\bullet(\underline{g}; M))$

Thus, we define

$$H^i_I(M) := H^i(\check{C}^\bullet(\underline{f}; M)) \text{ for } I = (\underline{f}).$$

i^{th} local cohomology of M with support in I.

If M is a D-module, then each $M_{f_{i_1} \cdots f_{i_t}}$ is a D-module, and each map $M_{f_{i_1} \cdots f_{i_t}} \xrightarrow{\pm 1} M_{f_{i_1} \cdots f_{i_t} f_{i_{t+1}}}$ is D-linear, so $\check{C}^\bullet(\underline{f}; M)$ is a complex of D-modules and each $H^i_I(M)$ is a D-module.

Will "compute" one of these soon.

Note that any left ideal of $D_{R|A}$ is a D-module, and any cyclic D-module is of the form $D_{R|A}/J$ for some left ideal J.

Left ideal gen. by $\delta_1, \dots, \delta_n$ is

$$\sum_i D_{R|A} \cdot \delta_i = \{\alpha_1\delta_1 + \dots + \alpha_n\delta_n \mid \alpha_i \in D_{R|A}\}.$$
$$\shortparallel$$
$$D_{R|A} \cdot \{\delta_i\}.$$

In general, we have

$$0 \to J \to D_{R|A} \xrightarrow[\text{evaluate at } 1]{} R \to 0$$

$$J = \{\delta \mid \delta(1) = 0\}.$$

as D-modules

As R-modules, this splits

$$D_{R|A} \rightleftarrows R, \quad r \mapsto r.$$

Sometimes this J is called the higher derivations or the differential operators.

(see more reading old papers)

For each i, have $D^0_{R|A} \subseteq D^i_{R|A}$, and this splits as R-modules:

$$0 \to \overline{D^i_{R|A}} \rightleftarrows D^i_{R|A} \rightleftarrows D^0_{R|A} \to 0$$

For $i=1$, $\overline{D^1_{R|A}}$ is the A-linear derivations on R, maps that ~~satisfy~~ satisfy Leibniz rule

$$\partial(xy) = x\,\partial(y) + y\,\partial(x).$$

(Exercise).

Let R be a poly ring over A,

We have $D_{R|A} = \bigoplus_{\alpha} R\,\partial^{(\alpha)}$.

Then $\ker\left(D_{R|A} \xrightarrow[\text{at } 1]{\text{evaluate}} R\right)$ is

$$J_1 = D_{R|A} \cdot \{\partial^{(\alpha)} \mid \alpha \neq \underline{0}\},$$ so

$R \cong D_{R|A}/J_1$.

In particular, if K is a field of char 0, then $J_1 = D_{R|K} \cdot \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$, and

$$R \cong D_{R|K} / D_{R|K} \cdot \{\tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_n}\}.$$

Let $J_2 := D_{R|A} \cdot \{\overline{x_1}, \dots, \overline{x_n}\}$ (R poly ring over A)

Write

$$H^n_{(\underline{x})}(R) = \frac{R_{x_1\cdots x_n}}{\sum_i R_{x_1\cdots \hat{x}_i \cdots x_n}}$$

$$\cong \frac{\bigoplus A\cdot\{\text{monomials } x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid \alpha_1,\dots,\alpha_n\in\mathbb{Z}\}}{\bigoplus A\cdot\{\text{monomials } x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid \alpha_1,\dots,\alpha_n\in\mathbb{Z} \text{ at least one } \alpha_i \text{ is nonnegative}\}}$$

$$\cong \bigoplus_{\alpha_1,\dots,\alpha_n<0} A\cdot x_1^{\alpha_1}\cdots x_n^{\alpha_n}.$$

Then $\overline{x_i}\cdot x_1^{\alpha_1}\cdots x_n^{\alpha_n} = \begin{cases} x_1^{\alpha_1}\cdots x_i^{\alpha_i+1}\cdots x_n^{\alpha_n} & \alpha_i<-1 \\ 0 & \alpha_i=-1\end{cases}$

and $\partial^{(\beta)}\cdot \underline{x}^{\alpha} = \underbrace{\binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n}}_{\text{always nonzero!}}\underline{x}^{\alpha-\beta}$

Then $\eta = x_1^{-1}\cdots x_n^{-1}$ is a generator for $H^n_{(\underline{x})}(R)$ as a D-module, and the annihilator of η is J_2 (exercise).

So $H^n_{(\underline{x})}(R) \cong D_{R|A}/J_2 = D_{R|A}/D_{R|A}\cdot\{\overline{x_1},\dots,\overline{x_n}\}$.

D-modules & differential equations

To any differential operator $\delta \in D_{\mathbb{R}[x]|\mathbb{R}}$ there is a differential equation

$(*)$ $\delta(f) = 0$.

$(\lambda - \frac{\partial}{\partial x})(f) = 0$.

$f = Ce^{\lambda x}$ 1-dim vs. $/\mathbb{R}$.

Likewise one can consider a linear system of PDE's:

$$(**) \quad \begin{bmatrix} \delta_{11} & \cdots & \delta_{1m} \\ \vdots & & \\ \delta_{n1} & \cdots & \delta_{nm} \end{bmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

We can express solving $(*)$ or $(**)$ as something algebraic.

Generally, people look for solutions in

$$\begin{cases} C^\infty(\mathbb{R}^n) \\ \mathbb{R}[[x_1, \dots, x_n]] \\ \mathbb{R}\{x_1, \dots, x_n\} \end{cases}$$

$\leftarrow$ functions analytic near $\underline{0}$

Remark: Each of these is a D-module.

Prop. For each of $M = \begin{cases} C^\infty(\mathbb{R}^n) \\ \mathbb{R}[[x_1, \dots, x_n]] \\ \mathbb{R}\{x_1, \dots, x_n\} \end{cases}$,

there is a bijection

$$\mathrm{Hom}_{\mathbb{R}[\underline{x}]|\mathbb{R}}\left(D_{\mathbb{R}[\underline{x}]|\mathbb{R}} / D_{\mathbb{R}[\underline{x}]|\mathbb{R}} \cdot \delta, M\right)$$

$$\updownarrow$$

$$\left\{ \text{solutions of } \delta(f) = 0 \text{ in } M \right\}$$

pf: Any $D_{\mathbb{R}[\underline{x}]|\mathbb{R}}$-linear map $\frac{D_{\mathbb{R}[\underline{x}]|\mathbb{R}}}{D_{\mathbb{R}[\underline{x}]|\mathbb{R}} \cdot \delta} \xrightarrow{\sigma} M$

is determined by the image of $\bar{1}$. Moreover, must have

$0 = \sigma(\bar{\delta}) = \sigma(\delta \bar{1}) = \delta \cdot \sigma(\bar{1})$, so $\sigma(\bar{1})$

must be a solution of $\delta(\sigma(\bar{1})) = 0$.

Conversely, if $\delta(f) = 0$, there is a map σ with $\sigma(\bar{1}) = f$. $\boxed{\equiv}$

Prop: For each M as above, there is a bijection

$$\mathrm{Hom}_{D_{\mathbb{R}[\underline{x}]|\mathbb{R}}}\left(\boxed{\mathrm{coker}\left(D^a_{\mathbb{R}[\underline{x}]|\mathbb{R}} \xrightarrow{A} D^b_{\mathbb{R}[\underline{x}]|\mathbb{R}}\right)}, M\right)$$

$$\updownarrow$$

$$\Big\}$$

$$\left\{ \text{solutions } (f_1, \dots, f_b) \text{ of } A \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_b \end{pmatrix} = \underline{0} \text{ in } M \right\}. \boxed{E}$$

Thus, every finitely presented D-module can be thought of as a linear system of PDEs.

(partial differential equation).

$$\mathbb{R}[x] = \frac{D_{\mathbb{R}[x]|\mathbb{R}}}{D_{\mathbb{R}[x]|\mathbb{R}} \cdot \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}} \longleftrightarrow \text{system } \frac{\partial}{\partial x_1}(f) = \dots = \frac{\partial}{\partial x_n}(f) = 0$$

constant functions.

$$H^n_{(x)}(\mathbb{R}[x]) = \frac{D_{\mathbb{R}[x]|\mathbb{R}}}{D_{\mathbb{R}[x]|\mathbb{R}} \cdot \{x_1, \dots, x_n\}} \longleftrightarrow x_1(f) = \dots = x_n(f) = 0$$

no solutions in M

$\Updownarrow$

Dirac δ-function.

$$H^i_{(x_1, \dots, x_n)}(A[x_1, \dots, x_n]) = \begin{cases} 0 & i \neq n \\ \bigoplus_{\alpha_1, \dots, \alpha_n < 0} A \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} & i = n. \end{cases}$$

$\delta \in D_{R|A}$

$\rightsquigarrow$

$M = D_{R|A} / D_{R|A} \cdot \delta$

δ probably not in R.