Last time:

Thm: K perfect field of char $p>0$
R ess. finite type $/K$
Then $D_{R|K} = \bigcup_{e\in\mathbb{N}} \mathrm{Hom}_{R^{p^e}}(R,R)$
and this filtration is "linearly cofinal" with the order filtration.

Each $\mathrm{Hom}_{R^{p^e}}(R,R)$ is a subring of $D_{R|K}$: often write $D_R^{(e)}$
"operators of level e."
Each $D_R^{(e)}$ is f.g. R-module,
since $D_R^{(e)} \subseteq D_{R|K}^{c p^e} \leftarrow$ f.g. R-mod.

Easy to see $D_{R|K}$ is not f.g. K-algebra in this setting, since any finite subset is contained in some $D_R^{(e)}$, so generates a subalgebra of $D_R^{(e)} \subsetneq D_{R|K}$.

For same reason,
if k has char 0, R poly ring over k, then $D_{R|k}$ has no filtration by subalgebras that is cofinal with the order filt.

<u>Ex:</u> Let $R = k[x_1, \dots, x_n]$ poly ring where k perfect of char $p>0$.
Then $R^{p^e} = k[x_1^{p^e}, \dots, x_n^{p^e}]$, and

$$R = \bigoplus_{0 \le \alpha_1, \dots, \alpha_n < p^e} R^{p^e} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \text{(free } R^{p^e}\text{-module)}.$$

(collect monomials via congruence classes of exponents modulo p^e).

Then $D_R^{(e)}$ is also a free R^{p^e}-module with basis $\left\{ \varphi_{\alpha,\beta} \;\middle|\; \begin{matrix} 0 \le \alpha_1, \dots, \alpha_n < p^e \\ 0 \le \beta_1, \dots, \beta_n < p^e \end{matrix} \right\}$

where $\varphi_{\alpha,\beta}(x_1^{\gamma_1} \cdots x_n^{\gamma_n}) = \begin{cases} x^\beta & \text{if } \underline{\gamma} = \underline{\alpha} \\ 0 & \text{otherwise} \end{cases}$

for $0 \le \gamma_1, \dots, \gamma_n < p^e$.

As a ring, $D_R^{(e)} \simeq \mathrm{Mat}_{p^{en} \times p^{en}}(R^{p^e})$.

~~That is~~, On the other hand,

$$D_{R|k} = \bigoplus_{\alpha} \overline{R}\, \partial^{(\alpha)}$$

So, can write $\frac{\partial}{\partial x_1}$ as a matrix, realizing it in $\mathrm{Hom}_{R^p}(R,R) \simeq \mathrm{Mat}_{p^n \times p^n}(R^p)$, or a different matrix in

$$\mathrm{Hom}_{R^{p^2}}(R,R) \simeq \mathrm{Mat}_{p^{2n}\times p^{2n}}(R^{p^2}).$$

$\vdots$

Conversely, any R^{p^e}-linear map from $R \to R$ can be written as an $\overline{R}$-linear combination of $\{\partial^{(\alpha)}\}$.

For example, consider

$\Phi := \varphi_{(p^e-1,\ldots,p^e-1),(0,\ldots,0)}$, R^{p^e}-linear map sending

$$x^{(p^e-1)\cdot \underline{1}} \mapsto 1$$
$$\text{others} \mapsto 0$$

We have that $\Phi = \partial^{(p^e-1,\ldots,p^e-1)}$.

To see it, write, $\underline{\alpha} = p^e \underline{\beta} + \underline{\gamma}$ with $0 \le \gamma_i < p^e$.

Then $\Phi(\underline{x}^{\underline{\alpha}}) = \underline{x}^{p^e \underline{\beta}}\, \Phi(\underline{x}^{\underline{\gamma}}) = \begin{cases} \underline{x}^{p^e \underline{\beta}} & \text{if each } \gamma_i = p^e-1 \\ 0 & \text{otherwise.} \end{cases}$

and $\partial^{(p^e-1,\ldots,p^e-1)}(\underline{x}^{\underline{\alpha}}) = \binom{\alpha_1}{p^e-1}\cdots\binom{\alpha_n}{p^e-1} x^{\underline{\alpha}-(p^e-1)\cdot\underline{1}}$

and check that $\binom{a}{p^e-1} \equiv \begin{cases} 1 \bmod p & \text{if } a \equiv p^e-1 \bmod p^e \\ 0 \bmod p & \text{if otherwise} \end{cases}$

(Exercise).

D-modules & D-simplicity

D-modules

A *D-module* is just a *left* $D_{R|A}$-module (for some understood $A \to R$).

We say that R-module M is a D-module if M is a left $D_{R|A}$-module and the given R-mod structure agrees with R-module str. on M via restriction of scalars via $R \longrightarrow D_{R|A}$; i.e.,

$$\underset{\substack{\uparrow \\ D_{R|A}\text{-action}}}{r \cdot m} = \underset{\substack{\uparrow \\ R\text{-action}}}{r \cdot m} \quad \text{for all } r \in R,\ m \in M.$$

$R = k[x_1, \dots, x_n]$, k char 0,

$D_{R|k} \curvearrowright R$

$\bar{x}_i$ mult by x_i

$\frac{\partial}{\partial x_i}$ differentiate by x_i.

Any ~~A~~ algebra R (any $A \to R$)

$\leadsto$ R is a D-module (a left $D_{R|A}$-module).

because $D_{R|A} \underset{\text{subring}}{\subseteq} \mathrm{End}_A(R)$.

Quotient rings?

~~Def~~ $R = S/I$ S poly ring over k

$$D_{R|k} \cong \frac{(I :_{D_{S|k}} I)}{I D_{S|k}}.$$

For R/I to be a $D_{R|A}$-module, we need any $\delta \in D_{R|A}$ to take I into I; that is: $(I :_{D_{R|A}} I) = D_{R|A}$.

We saw that if $R = S/I$ Stanley-Reisner ring (S poly ring), $I = P_1 \cap \cdots \cap P_t$, P_i monomial primes then each $(P_i :_{D_{R|A}} P_i) = D_{R|A}$.

Thus, in this case, each P_i or each R/P_i is a D-module.

<u>Def:</u> $A \to R$ commut. ring

A <u>D-ideal</u> is a D-submodule of R.

Since $R \hookrightarrow D_{R|A}$, a D-ideal is an ideal of R.

<u>Lem:</u> Let I, J be D-ideals. Then $I+J$, $I \cap J$, and every minimal primary component of I (if R is Noeth.) is a D-ideal.

<u>pf:</u> For "$+$" and "$\cap$", generally fact about submodules.

If Q is a min'l primary cpt. of I, then

$$(Q :_{D_{R|A}} Q) = (I :_{D_{R|A}} I) = D_{R|A}.$$

∎

Rmk: If I is a D-ideal, R/I is a D-module.

The other big source of D-modules is localization.

$D_{R|A} \rightsquigarrow D_{W^{-1}R|A}$

Prop: Let $A \to R$ comm. rings. Let M be a D-module. Let $W \subseteq R$ ~~be~~ mult. closed. Then $W^{-1}M$ is a D-module ($D_{R|A}$-module) by the rule

$$\boxed{\alpha \cdot \frac{m}{w} = \sum_{i=0}^{\operatorname{ord}(\alpha)} (-1)^i \frac{\alpha^{(i)} \cdot m}{w^{i+1}}}$$

where $\alpha^{(0)} := \alpha$, $\alpha^{(i+1)} := [\alpha^{(i)}, \bar{w}]$.

Pf: We will use $?\circledast?$ to denote the function $D_{R|A} \times W^{-1}M \xrightarrow{\circledast} W^{-1}M$ defined inductively on order of elt in $D_{R|A}$ by rule

$$\boxed{\alpha \circledast \frac{m}{w} := \frac{1}{w}\left(\alpha \cdot m - [\alpha, \bar{w}] \circledast {}^{m}/_{w}\right)},$$

where $? \cdot ?$ denotes the given action $D_{R|A} \times M \xrightarrow{\cdot} M$.

Then $\circledast$ is well-defined, and A-bilinear.

By a ~~straightforward~~ induction, $\circledast$ agrees with the action we defined in the statement.

Will write $\alpha' := [\alpha, \bar{w}]$.

Observe first that if $\bar{r} \in D^0_{R|A}$, then $\bar{r}$ acts by mult. on $W^{-1}M$.

To check

$$\alpha \circledast \left(\beta \circledast \frac{m}{w}\right) = (\alpha \circ \beta) \circledast \frac{m}{w}.$$

(i) Case $\alpha \in D^0_{R|A}$:

$\alpha = \bar{r} \rightsquigarrow$

$$\bar{r} \circledast \left(\beta \circledast \frac{m}{w}\right) = \frac{r}{w}\left(\beta \cdot m - \beta' \circledast \frac{m}{w}\right)$$

$$= \frac{1}{w}\left(r(\beta \cdot m) - r\left(\beta' \circledast \frac{m}{w}\right)\right) = \frac{1}{w}\left(\bar{r} \cdot (\beta \cdot m) - \bar{r} \circledast \left(\beta' \circledast \frac{m}{w}\right)\right)$$

M is a D-module

Induction on order of β.

Aside: why this formula? If $\ast$ worked

$$\alpha \ast \left(w \frac{m}{w}\right) = \alpha \ast m = \alpha \cdot m$$

$$= \alpha \bar{w} \ast \frac{m}{w} = (\bar{w}\alpha + [\alpha, \bar{w}]) \ast \frac{m}{w} = (w\alpha) \ast \frac{m}{w} + [\alpha, \bar{w}] \ast \frac{m}{w}$$

$$\alpha \ast \frac{m}{w} = \frac{1}{w}\left(\alpha \cdot m - [\alpha, \bar{w}] \ast \frac{m}{w}\right)$$

$$= \frac{1}{w}\big((\bar{r}\circ\beta)\bullet m - (\bar{r}\circ\beta')\circledast \tfrac{m}{w}\big) = (\bar{r}\circ\beta)\circledast m/w.$$

(ii) Case $\beta \in P^{0}_{RIA}$: similar.

(iii) General case, by induction on $\mathrm{ord}(\alpha)+\mathrm{ord}(\beta)$:

To show: $w(\alpha\circledast(\beta\circledast \tfrac{m}{w})) \overset{w}{=} (\alpha\circ\beta)\circledast\tfrac{m}{w}$

Definition: $\alpha * \frac{m}{w} = \frac{1}{w}\big(\alpha\bullet m - [\alpha,\bar{w}] * \frac{m}{w}\big)$

$$\underline{RHS} = (\alpha\circ\beta)\bullet r - (\alpha\circ\beta)'\circledast(r/w)$$

rule on brackets

$$= \alpha\bullet(\beta\bullet r) - (\alpha\circ\beta')\circledast(r/w) - (\alpha'\circ\beta)\circledast(r/w)$$

IH on order

$$= \alpha\bullet(\beta\bullet r) - \alpha\circledast(\beta'\circledast r/w) - \alpha'\circledast(\beta\circledast r/w)$$

$$\underline{LHS} = w(\alpha\circledast(\beta\circledast r/w)) \overset{(i)}{=} (\bar{w}\alpha)\circledast(\beta\circledast r/w)$$

$$= (\alpha\bar{w} - \alpha')\circledast(\beta\circledast r/w)$$

$$= \alpha\bar{w}\circledast(\beta\circledast(r/w)) - \alpha'\circledast(\beta\circledast r/w)$$

(i) & (ii)

$$= \alpha\circledast(\bar{w}\beta)\circledast r/w - \text{same}$$

$$= \alpha\circledast(\beta\bar{w} - \beta')\circledast r/w - \text{———}''$$

$$= \alpha\circledast\beta\bar{w}\circledast r/w - \alpha\circledast\beta'\circledast r/w - \text{———}''$$

$$= \alpha\circledast\beta\circledast\bar{w}\, r/w - \text{———}'' - \text{———}''$$

$$= \alpha\circledast\beta\circledast r - \text{———}'' - \text{———}''$$

$$= \alpha\bullet\beta\bullet r - \text{———}'' - \text{———}''. \quad \boxed{\varepsilon}$$