Differential operators on Stanley-Reisner rings

Our next class of examples is Stanley-Reisner rings. Let $S = k[x_1, \dots, x_n]$ be a poly. ring over a field k, $I = P_1 \cap \cdots \cap P_n$ ~~be~~ a squarefree monomial ideal, where $P_i = (\{x_j \mid j \in S_i\})$ are monomial prime ideals. $R = S/I$ is a Stanley-Reisner ring.

First, we observe a fact about operators preserving ideals.

<u>Prop</u>: Let $A \to R$ be comm. rings, with R Noetherian. ~~Let~~ $I \subseteq R$ be an ideal, and Q a minimal primary component of I. Then

1) $\delta \in (I :_{D_{R|A}} I) \Rightarrow [\delta, \bar{r}] \in (I :_{D_{R|A}} I)$ for any $r \in R$.
2) $(I :_{D_{R|A}} I) \subseteq (Q :_{D_{R|A}} Q)$.

<u>Pf</u>: 1) If $a \in I$, then $[\delta, \bar{r}](a) = \delta(ra) - r\delta(a) \in I$.

2) Let f be in the intersection of the other primary components of I but outside of $\sqrt{Q}$.

If the $\cap$ of the other components is contained in $\sqrt{Q}$ then some other cpt. is ctd. in $\sqrt{Q}$, contradicting that its unique minimal prime isn't ctd in $\sqrt{Q}$.

Note that $fQ \subseteq I$.

We will show that $(I :_{D^i_{R|A}} I) \subseteq (Q :_{D^i_{R|A}} Q)$ by induction on i, with base case $i=0$ trivial.

$[\delta, \bar{f}] \in (Q :_{D^{i-1}_{R|A}} Q)$ by Part (i) and IH.

For $q \in Q$, then, $\delta(fq) - f\delta(q) \in Q$.

$fq \in I \Rightarrow \delta(fq) \in I \subseteq Q$, so $f\delta(q) \in Q$, hence $\delta(q) \in Q$. $\square$

Prop: Let Q be a monomial prime. Then $(Q :_{D_{S|K}} Q) = K\cdot\{x^{\alpha}\partial^{(\beta)} \mid x^{\alpha}\in Q \text{ or } x^{\beta}\notin Q\}$.

prf: For the containment $(\supseteq)$, it suffices to check for the basis elements.

If $x^{\alpha}\in Q$, $x^{\alpha}\partial^{(\beta)}\in(Q :_{D_{S|K}} Q)$ is clear.

If $x^{\beta}\notin Q$, then $x^{\alpha}\partial^{(\beta)}$ can only decrease exponents of the variables not in Q, so must stabilize Q.

For the other containment, suppose $\delta\in(Q :_{D_{S|K}} Q)$ is not in the RHS. Subtracting off an element of the RHS, we can assume that all of the terms $\lambda_{\alpha,\beta}x^{\alpha}\partial^{(\beta)}$ $\lambda_{\alpha,\beta}\neq 0$ have $x^{\alpha}\notin Q$ and $x^{\beta}\in Q$.

Let $(\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t)$ be the nonzero pairs (α,β) as above in δ. Suppose that β_1 is such that $|\beta_1|\leq|\beta_i|$ all i.

Then $\delta(x^{\beta_1}) = \sum_{(\alpha,\beta)} x^{\alpha}\partial^{(\beta)}(x^{\beta_1})$

$$= x^{\alpha_1}\partial^{(\beta_1)}(x^{\beta_1}) = x^{\alpha_1},$$

but $x^{\beta_1}\in Q$, $x^{\alpha}\notin Q$ contradicts that $\delta\in(Q :_{D_{S|K}} Q)$. ∎

<u>Thm</u> (Tripp, Traves):

Let $R = S/I$ be a Stanley-Reisner ring, with $I = P_1 \cap \cdots \cap P_t$ squarefree monomial ideal, P_i primes. Then

$$D_{R|K} = K\left\{ x^{\alpha}\partial^{(\beta)} \;\middle|\; x^{\alpha} \notin I,\ x^{\alpha} \in P_i \text{ or } x^{\beta} \notin P_i \text{ for each } i \right\}$$

as K-vectorspaces, with composition induced by the corresponding operators on S.

Pf: We use the description

$$D_{R|K} = (I :_{D_{S|K}} I) / I D_{S|K}.$$

To compute $(I :_{D_{S|K}} I)$, observe first that if $\delta \in \bigcap_i (P_i :_{D_{S|K}} P_i)$, and $a \in I$, then $a \in P_i$ for each i, so $\delta(a) \in P_i$ for each i, hence $\delta(a) \in I$. Conversely, $(I :_{D_{S|K}} I) \subseteq \bigcap_i (P_i :_{D_{S|K}} P_i)$ by Prop above, and the equality holds.

Using the previous proposition,

$$(P_i :_{D_{S|K}} P_i) = K\{ x^{\alpha}\partial^{(\beta)} \mid x^{\alpha} \in P_i \text{ or } x^{\beta} \notin P_i \}.$$

Then the given basis above comes from intersecting these and removing those in $I D_{S|K}$ in the monomial basis. $\square$

Example: Let $R = K[x,y]/(xy)$.

Writing $(xy) = (x) \cap (y)$, we have

$$D_{R|K} = K \cdot \left\{ \bar{x}^a \bar{y}^b \partial^{(c,d)} \;\middle|\; \begin{array}{l} a \text{ or } b = 0 \\ a>0 \text{ or } c=0 \\ b>0 \text{ or } d=0 \end{array} \right\}$$

$$= K \cdot \{\bar{1}\} \cup \left\{ \bar{x}^a \partial^{(c,0)} \;\middle|\; \begin{array}{l} a \geq 1 \\ c \geq 0 \end{array} \right\} \cup \left\{ \bar{y}^b \partial^{(0,d)} \;\middle|\; \begin{array}{l} b \geq 1 \\ d \geq 0 \end{array} \right\}.$$

If K has characteristic zero, then we write as

$$K \cdot \left\{ \bar{1}, \bar{x}^a \left(\frac{\partial}{\partial x}\right)^c \begin{array}{l} a \geq 1 \\ c \geq 0, \end{array} \bar{y}^b \left(\frac{\partial}{\partial y}\right)^d \begin{array}{l} b \geq 1 \\ d \geq 0 \end{array} \right\}.$$

Even in the char 0 case, $D_{R|K}$ is not a finitely gen. K-algebra: in every order i, there is $\bar{x}\left(\frac{\partial}{\partial x}\right)^i \in D^i_{R|K}$ that is not in the algebra generated by $D^{i-1}_{R|K}$; thus any algebra gen. set must involve arbitrarily high orders, and thus must be infinite.

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Cone over elliptic curve

~~Def~~ $R = \dfrac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$.

Then [Bernstein-Gelfand-Gelfand]:

- $[D_{R|\mathbb{C}}]_{<0} = 0$: there are no differential operators of negative degree
- $[D_{R|\mathbb{C}}]_0 = \mathbb{C}[\underbrace{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}}_{E}]$: every operator of degree zero is a poly in the Euler operator
- $\dfrac{[D^i_{R|\mathbb{C}}]_1}{[D^{i-1}_{R|\mathbb{C}}]_1 + E\cdot[D^{i-1}_{R|\mathbb{C}}]_1} \simeq \mathbb{C}^3$ for each i.

It follows from these facts that $D_{R|\mathbb{C}}$ is not a finitely generated $\mathbb{C}$-algebra. Indeed, if we set

$$A_k = [D_{R|\mathbb{C}}]_0 + \sum_{n\geq 0} E^n [D^k_{R|\mathbb{C}}]_1 + [D_{R|\mathbb{C}}]_{\geq 2},$$

then $D^k_{R|\mathbb{C}} \subseteq A_k \subsetneq A_{k+1} \subseteq D_{R|\mathbb{C}}$ for each k,

~~so~~ $D_{R|\mathbb{C}}$ is not generated by $D^k_{R|\mathbb{C}}$ for any k, ~~so~~ is ~~not~~ fin. generated.

We skip the proof of this: one proceeds by analyzing the cohomology of the tangent bundle on Proj(R) the curve.

7 Differential operators in positive characteristic.

<u>Theorem</u>: Let K be a perfect field, and R be essentially of finite type over K. Then

$$D_{R|K} = \bigcup_{e\in\mathbb{N}} \mathrm{Hom}_{R^{p^e}}(R,R)$$

and there is a constant C such that

$$D^{p^e}_{R|K} \subseteq \mathrm{Hom}_{R^{p^e}}(R,R) \subseteq D^{Cp^e}_{R|K}$$

for all $e \geq 0$.

<u>pf</u>: First we observe that $K = K^{p^e} \subseteq R^{p^e}$ for each e, so $\mathrm{Hom}_{R^{p^e}}(R,R) \subseteq \mathrm{Hom}_K(R,R)$ for each e, and both sides of the equality can <u>be</u> considered as subsets of $\mathrm{Hom}_K(R,R)$.

Now, $R\otimes_K R$ is ess. fin. type over K, hence Noeth., so $\Delta_{R|K}$ is finitely generated, say with C generators. Then $\Delta^{Cp^e}_{R|K} \subseteq \Delta^{[p^e]}_{R|K} \subseteq \Delta^{p^e}_{R|K}$ for each e by the pigeonhole principle. Thus, we have

$$\begin{array}{ccccc} (0:_{\mathrm{Hom}_K(R,R)} \Delta^{Cp^e}_{R|K}) & \supseteq & (0:_{\mathrm{Hom}_K(R,R)} \Delta^{[p^e]}_{R|K}) & \supseteq & (0:_{\mathrm{Hom}_K(R,R)} \Delta^{p^e}_{R|K}) \\ \| & & \| & & \| \\ D^{Cp^e}_{R|K} & \supseteq & \mathrm{Hom}_{R^{p^e}}(R,R) & \supseteq & D^{p^e}_{R|K} \end{array}$$

where the middle equality comes from the observation ~~that~~

$$\Delta^{[p^e]}_{R|K} = (\{1\otimes r^{p^e} - r^{p^e}\otimes 1 \mid r\in R\}) = (\{1\otimes s - s\otimes 1 \mid s\in R^{p^e}\})$$

$= \Delta_{R^{p^e}|K}(R\otimes_K R)$, so this ideal kills a map $\Leftrightarrow$ the map is R^{p^e}-linear. $\boxtimes$