Prop: Let $k=\bar{k}$. If G contains no pseudoreflections, then the restriction map $D^n_{R|k} \xrightarrow{\beta} D^n_{R^G|k}(R^G,R)$ is an isomorphism.

pf: Note first that $D^n_{R|k}$ is a free R-module, and that $D^n_{R^G|k}(R^G,R) \simeq \mathrm{Hom}_{R^G}(P^n_{R^G|k},R)$ is a torsionfree R-module, since postmultiplying by an element in R, which is torsionfree, cannot kill a nonzero map.

We claim now that if $\mathrm{ht}(\mathfrak{p}) \leq 1$ in R, then $\beta_{\mathfrak{p}}$ is an isomorphism. To see this, note that the hypothesis means that $V(\mathfrak{p}) \nsubseteq \bigcup_{\sigma \neq e} \mathrm{Fix}(\sigma)$. As $\mathfrak{p}$ is an intersection of maximal ideals, $\exists\, \mathfrak{m} \in V_{\max}(\mathfrak{p}) \setminus \bigcup_{\sigma \neq e} \mathrm{Fix}(\sigma)$. That is, there is a max'l ideal $\mathfrak{m}$ with $\mathfrak{m} \supseteq \mathfrak{p}$ and $\mathrm{Stab}(\mathfrak{m}) = \{e\}$. Thus, $\beta_{\mathfrak{m}}$ is an isomorphism, and localizing further, $\beta_{\mathfrak{p}}$ is an isomorphism as well.

To see that β is injective, we have $\mathrm{Ass}_R(\ker(\beta)) \subseteq \mathrm{Ass}_R(D^n_{R|K}) = (0)$, but $\beta_{(0)}$ is injective, so $\mathrm{Ass}(\ker(\beta)) = \emptyset$, and β is injective.

Since $\mathrm{Hom}_{R^G}(P^n_{R^G|K}, R)$ is torsion-free, $\operatorname{depth}_{R_\mathfrak{p}}(\mathrm{Hom}_{R^G}(P^n_{R^G|K}, R)_\mathfrak{p}) \geq 1$.
For all primes $\mathfrak{p} \neq (0)$:

Then if $\mathrm{ht}(\mathfrak{p}) \leq 1$, $\mathrm{coker}(\beta)_\mathfrak{p} = 0$, and if $\mathrm{ht}(\mathfrak{p}) \geq 2$, $\operatorname{depth}((D^n_{R|K})_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) \geq 2$, since $(D^n_{R|K})_\mathfrak{p}$ is free over $R_\mathfrak{p}$ (which is Cohen-Macaulay). Thus, if $\mathrm{coker}(\beta)_\mathfrak{p} \neq 0$, by the behavior of depth on SES's, we have $\operatorname{depth}(\mathrm{coker}(\beta)_\mathfrak{p}) \geq 1$. But, if $\mathrm{coker}(\beta) \neq 0$, $\exists\, q \in \mathrm{Ass}(\mathrm{coker}(\beta))$, and $\operatorname{depth}(\mathrm{coker}(\beta)_q) = 0$. Thus, $\mathrm{coker}(\beta) = 0$, so β is an isomorphism. $\boxtimes$

7 Prop: Without assuming $k=\bar{k}$, if G has no pseudoreflections then the restriction map $D^n_{R|k} \xrightarrow{\beta} D^n_{R^G|k}(R^G,R)$ is an isomorphism.

prf: It suffices to show that

$$D^n_{R|k} \otimes_k \bar{k} \xrightarrow{\beta \otimes_k \bar{k}} D^n_{R^G|k}(R^G,R) \otimes_k \bar{k}$$

is an isomorphism, since $k \to \bar{k}$ is faithfully flat.

Let $R_{\bar{k}} = R \otimes_k \bar{k}$, poly ring over $\bar{k}$.
Then G acts on $R_{\bar{k}}$ by $\sigma(r \otimes \lambda) := \sigma(r) \otimes \lambda$.
Observe that the action of G on R has no pseudoreflections if and only if the action of G on $R \otimes_k \bar{k}$ ~~does~~ has none since the rank of $(\mathrm{id}-\sigma)$ on 1-forms is the same.

We will show that $\beta \otimes_k \bar{k}$ identifies with

$$D^n_{R_{\bar{k}}|\bar{k}} \xrightarrow{\beta_{\bar{k}}} D^n_{(R_{\bar{k}})^G|\bar{k}}((R_{\bar{k}})^G, R_{\bar{k}}),$$

and then we will be done by the previous proposition.

First, note that we have a left-exact sequence

$$0 \to R^G \to R \xrightarrow{\begin{bmatrix} \mathrm{id}-\sigma_1 \\ \vdots \\ \mathrm{id}-\sigma_{|G|} \end{bmatrix}} R, \quad G = \{\sigma_1, \dots, \sigma_{|G|}\},$$

so by flatness we have

$$0 \to R^G \otimes_k \bar{k} \to R_{\bar{k}} \xrightarrow{\begin{bmatrix} \mathrm{id}-\sigma_1 \\ \vdots \\ \mathrm{id}-\sigma_{|G|} \end{bmatrix}} R_{\bar{k}}, \text{ so } R^G \otimes_k \bar{k} \cong (R_{\bar{k}})^G$$

canonically. Then, if S is any f.g. k-algebra, we have $P^n_{S|k} \otimes_k \bar{k} \cong P^n_{S\otimes \bar{k}|\bar{k}}$, and by the behavior of Hom & flat base change, and finite presentation of $P^n_{S|k}$, we obtain $D^n_{S|k} \otimes_k \bar{k} \cong D^n_{S\otimes\bar{k}|\bar{k}}$.

Applying these two observations, we obtain the desired identification. ☐

Theorem (Kantor): Let G be a finite group acting linearly on a polynomial ring R over a field K, with $|G| \neq 0$ in K. Assume G contains no pseudoreflections. Then the restriction map

$$\{\delta \in D^n_{R|K} \mid \delta(R^G) \subseteq R^G\} \longrightarrow D^n_{R^G|K}$$ is an isomorphism. Moreover, we have the equality

$$\{\delta \in D^n_{R|K} \mid \delta(R^G) \subseteq R^G\} = (D^n_{R|K})^G, \text{ where}$$

G acts on $D^n_{R|K}$ via $g \cdot \delta = g \circ \delta \circ g^{-1}$.

There are maps

$$D^n_{R^G|k} \xrightarrow{i} D^n_{R^G|k}(R^G,R) \xrightarrow{\sim} D^n_{R|k}.$$

The second map has inverse $\leftarrow$ given by restriction. The first is injective, since there is an inverse $\leftarrow$ $\delta \mapsto \frac{1}{|G|}\sum_{g\in G} g\circ\delta$. We see that the composition $\to\to$ has image equal to the maps sending R^G into R^G, and any such map goes via $\leftarrow\leftarrow$ to this restriction.

For the second claim, if $\delta\in (D^n_{R|k})^G$, and $r\in R^G$, then $g(\delta(r)) = (g\cdot\delta)(g(r)) = \delta(r)$, so $\delta(r)\in R^G$, thus $(D^n_{R|k})^G \subseteq \{\delta\in D^n_{R|k} \mid \delta(R^G)\subseteq R^G\}$. For the other containment, let $\delta(R^G)\subseteq R^G$, and take $g\in G$. We need to show that $g\cdot\delta-\delta$ is zero in $D^n_{R|k}$. By the second BO. above, it suffices to show it is zero on R^G, so let $r\in R^G$. Then $(g\cdot\delta-\delta)(r) = g\delta(g^{-1}(r)) - \delta(r) = g\delta(r)-\delta(r)=0$, since $\delta(r)\in R^G$. ∎

Example: Let K be an algebraically closed field, $R = K[x_1, \dots, x_n]$, $n \geq 2$, and d be an integer ~~that~~ is invertible in K (not a multiple of char K, if char $K = p > 0$).

$G \simeq \mathbb{Z}/d$ acts on R by $g \cdot x_i = \zeta x_i$ for g a generator of G, and ζ a primitive d^{th} root of unity.

The fixed space of g, or any nonidentity element, is just the origin, so the theorem applies.

We have $R^G = R^{(d)} := \bigoplus_{n \in \mathbb{N}} [R]_{nd}$, the d^{th} Veronese subring, which consists of elements whose homogeneous pieces have degrees a multiple of d.

We compute

$$D_{R^G} = \{ \delta \in D_R \mid \delta(R^G) \subseteq R^G \} \text{ (restrictions of)}$$

Write any $\delta \in D_R$ as a sum of homogeneous pieces $\delta = \sum_j \delta_j$. Then δ_j of degree j satisfies $\delta_j(R^G) \subseteq R^G \iff d \mid j$. We conclude that

$$D_{R^G} = \{ \delta \in D_R \mid \text{every homogeneous piece of } \delta \text{ has degree a multiple of } d \}.$$

For example, $D_{R^{(2)}} = R \langle \{ x_i \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \} \rangle$ if char $K = 0$.

Exercise: Suppose that $g \in G$ acts on R via
$$g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$
Then the action of g on D_R is given by (char $k=0$)
$$g \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix} = \begin{pmatrix} A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ (A^T)^{-1} \cdot \begin{pmatrix} \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix} \end{pmatrix}.$$