Invariants of finite groups

Our next family of examples is invariants of finite groups. Throughout, the setting is as follows:

K: field

$R = K[x_1, \dots, x_n]$ poly ring

G finite group, with order $|G|$ nonzero in K, acting linearly on R.

$\mathrm{Spec}(R) \supseteq \mathrm{Max}(R) \supseteq \mathrm{Max}_K(R)$

Note that G acts on $\mathrm{Spec}(R)$: $\sigma(\mathfrak{p})$ is a (maximal) prime for $\mathfrak{p}$ (maximal) prime $\sigma \in G$.
Thus, for $\sigma \in G$ we have $\mathrm{Fix}(\sigma) \subseteq \mathrm{Max}(R)$ and for $\mathfrak{p} \in \mathrm{Max}(R)$ we have $\mathrm{Stab}(\mathfrak{p}) \leq G$.

Prop: Let $K = \overline{K}$. Let $\mathfrak{n} \in \mathrm{Max}(R)$ be s.t. $\mathrm{Stab}(\mathfrak{n}) = \{e\}$, and let $\mathfrak{m} = \mathfrak{n}^G = \mathfrak{n} \cap R^G$. Then
$R^G/\mathfrak{m}^n \cong R/\mathfrak{n}^n$ for all $n \in \mathbb{N}$.

Prf: Since R^G and R are f.g. algebras over an alg. closed field, we have $\mathfrak{m} \in \mathrm{Max}(R^G)$, and $R^G/\mathfrak{m} \cong K \cong R/\mathfrak{n}$.

We claim that $\mathfrak{m}R_{\mathfrak{n}} = \mathfrak{n}R_{\mathfrak{n}}$. To see this, let $J = (\{f \in \mathfrak{n} \mid f \notin \sigma(\mathfrak{n}) \ \forall \sigma \neq e\})$.
Then $\mathfrak{n} = J \cup \left(\bigcup_{\sigma \neq e} \sigma(\mathfrak{n})\right)$.

Each $\sigma(\mathfrak{n})$ is a maximal ideal distinct from $\mathfrak{n}$ (by $Stab(\mathfrak{n}) = \{e\}$ hypothesis), so $\mathfrak{n} \not\subseteq \sigma(\mathfrak{n})$, and by prime avoidance, we must have $\mathfrak{n} \subseteq J$, so $\mathfrak{n} = J$. Thus, we can write $\mathfrak{n} = (f_1, \dots, f_t)$ with $\sigma(f_i) \notin \mathfrak{n}$ for each $\sigma \neq e$, each i. Then $g_i = \prod_{\sigma \neq e} \sigma(f_i) \notin \mathfrak{n}$, so g_i are units in $R_{\mathfrak{n}}$. Thus, $\mathfrak{n} = (g_1 f_1, \dots, g_t f_t)$, and each $g_i f_i \in R^G \cap \mathfrak{n} = \mathfrak{m}$, fulfilling the claim. [Exercise: The equality $\mathfrak{m}R = \bigcap_{\sigma \in G} \sigma(\mathfrak{n})$ holds in this setting.]

Now the map $R^G/\mathfrak{m}^n \to R/\mathfrak{n}^n$ is module-finite (the target is even finite length) so the cokernel $M = R_{\mathfrak{n}}/(\mathrm{im}(R^G) + \mathfrak{n}^n)$ is as well. But $M/\mathfrak{m}M \cong R_{\mathfrak{n}}/(\mathfrak{m}R_{\mathfrak{n}} + \mathrm{im}(R^G) + \mathfrak{n}^n) \cong 0$, so $M = 0$ by NAK. Thus the map is surjective.

For injectivity, we'll need to show that $\mathfrak{n}^n \cap R^G \subseteq \mathfrak{m}^n$. Note that if $I = (a_1, \dots, a_s) \subseteq R$ and $f \in I \cap R^G$, we have $f = \sum a_i r_i \rightsquigarrow f = \sigma(f) = \sum \sigma(a_i)\sigma(r_i) \in \sigma(I)$ for $\sigma \in G$. Thus, $\mathfrak{n}^n \cap R^G \subseteq \bigcap_{\sigma \in G} \sigma(\mathfrak{n}^n) = \prod_{\sigma \in G} \sigma(\mathfrak{n}^n)$ (by Chinese Remainder Theorem)

$$= \prod_{\sigma \in G} \sigma(\mathfrak{n})^n = \Big(\prod_{\sigma \in G} \sigma(\mathfrak{n})\Big)^n = \Big(\bigcap_{\sigma \in G} \sigma(\mathfrak{n})\Big)^n = (\mathfrak{m}R)^n$$

$$= \mathfrak{m}^n R, \quad \text{so } \mathfrak{n}^n \cap R^G \subseteq \mathfrak{m}^n R \cap R^G = \mathfrak{m}^n,$$

since R^G is a direct summand of R. ∎

There is a natural map

$$\begin{array}{ccc} R \otimes_{R^G} P^n_{R^G|K} & \simeq & \dfrac{R \otimes_K R^G}{\Delta^n_{R^G|K}(R \otimes_K R^G)} \\ \alpha \downarrow & & \downarrow \\ P^n_{R|K} & \simeq & \dfrac{R \otimes_K R}{\Delta^n_{R|K}} \end{array}$$

, since $\Delta_{R^G|K}(R \otimes_K R) \subseteq \Delta_{R|K}$.

<u>Prop</u>: Let $K=\bar{K}$. ~~Let~~ $\mathfrak{n} \in Max(R)$ be s.t. $stab(\mathfrak{n}) = \{e\}$, and ~~let~~ $\mathfrak{m} = \mathfrak{n} \cap R^G$. Then the localization $\alpha_{\mathfrak{n}}: R_{\mathfrak{n}} \otimes_{R^G_{\mathfrak{m}}} P^n_{R^G_{\mathfrak{m}}|K} \longrightarrow P^n_{R_{\mathfrak{n}}|K}$ is an isomorphism.

prf: ~~First~~, we check surjectivity. Note that $P^n_{R_{\mathfrak{n}}|K}$ is finitely generated, so by NAK, it suffices to see ~~that~~ $P^n_{R_{\mathfrak{n}}|K} = im(\alpha_{\mathfrak{n}}) + \mathfrak{n} P^n_{R_{\mathfrak{n}}|K}$. ~~But~~,

$$\frac{P^n_{R_{\mathfrak{n}}|K}}{\mathfrak{n} P^n_{R_{\mathfrak{n}}|K}} \simeq K \otimes_{R_{\mathfrak{n}}} P^n_{R_{\mathfrak{n}}|K} \simeq \frac{R_{\mathfrak{n}}}{\mathfrak{n}^{n+1}} \simeq \frac{R^G_{\mathfrak{m}}}{\mathfrak{m}^{n+1}} \simeq \frac{R_{\mathfrak{n}} \otimes_{R^G_{\mathfrak{m}}} P^n_{R^G_{\mathfrak{m}}|K}}{\mathfrak{n}(R_{\mathfrak{n}} \otimes_{R^G_{\mathfrak{m}}} P^n_{R^G_{\mathfrak{m}}|K})},$$

verifying surjectivity.

To see injectivity, note that $P^n_{R_{\mathfrak{n}}|K}$ is local with maximal ideal $(\mathfrak{n} \otimes 1 + 1 \otimes \mathfrak{n})$:

$Spec\left(\frac{R_{\mathfrak{n}} \otimes_K R_{\mathfrak{n}}}{\Delta^{n+1}_{R_{\mathfrak{n}}|K}}\right) \cong Spec\left(\frac{R_{\mathfrak{n}} \otimes_K R_{\mathfrak{n}}}{\Delta_{R_{\mathfrak{n}}|K}}\right) \simeq Spec(R_{\mathfrak{n}})$, which is local;

similarly, $R_{\mathfrak{n}} \otimes_{R^G_{\mathfrak{m}}} P^n_{R^G_{\mathfrak{m}}|K}$ is local w/max ideal $(\mathfrak{n} \otimes 1 + 1 \otimes \mathfrak{m})$.

Let α_t be the map $\dfrac{R_{\mathfrak{n}} \otimes_{R^G_{\mathfrak{m}}} P^n_{R^G_{\mathfrak{m}}|K}}{(\mathfrak{n}^t \otimes 1 + 1 \otimes \mathfrak{m}^t)} \longrightarrow \dfrac{P^n_{R_{\mathfrak{n}}|K}}{(\mathfrak{n}^t \otimes 1 + 1 \otimes \mathfrak{n}^t)}$.

An element in $R_{\mathfrak{n}} \otimes_{R^G_{\mathfrak{m}}} P^n_{R^G_{\mathfrak{m}}|K}$ is in $ker(\alpha_{\mathfrak{n}}) \Longleftrightarrow$ its image is in $ker(\alpha_t)$ for each t, since $\bigcap_t (\mathfrak{n}^t \otimes 1 + 1 \otimes \mathfrak{n}^t) \subseteq \bigcap_t (\mathfrak{n} \otimes 1 + 1 \otimes \mathfrak{n})^t = 0$, by Krull-$\cap$.

But the isomorphism $R^G/\mathfrak{n}^t \simeq R_\mathfrak{m}/\mathfrak{n}^t$ induces isomorphisms α_t for every t. Thus,
$\ker(\alpha_\mathfrak{m}) \subseteq \bigcap_t (\mathfrak{n}^t \otimes 1 + 1 \otimes \mathfrak{m}^t) \subseteq \bigcap_t (\mathfrak{n} \otimes 1 + 1 \otimes \mathfrak{m})^t = 0$,
so $\alpha_\mathfrak{m}$ is also injective. $\boxed{\Xi}$

Applying $\mathrm{Hom}_R(-,R)$, α induces a map

$$\begin{tikzcd}
\mathrm{Hom}_R(P^n_{R/K}, R) \arrow[r] \arrow[d, "\wr", "d^*"'] & \mathrm{Hom}_R(R \otimes_{R^G} P^n_{R^G/K}, R) \simeq \mathrm{Hom}_{R^G}(P^n_{R^G/K}, R) \arrow[d, "\wr", "d^*"'] \\
D^n_{R/K} \arrow[r, "\beta"] & D^n_{R^G/K}(R^G, K)
\end{tikzcd}$$

<u>Exercise</u>: β is just the restriction map.
It follows that $\beta_\mathfrak{m}$ is an isomorphism for all $\mathfrak{m} \in \mathrm{Max}(R)$ with trivial stabilizer.

discuss at beginning

Let $\sigma \in G$. Consider $\mathrm{Max}_K(R) \simeq K^n$ to be the maximal ideals of R with residue field $\simeq K$; these are all of the form
$\mathfrak{m}_\alpha = (x_1 - \alpha_1, \ldots, x_n - \alpha_n)$, for $\alpha \in K^n$.
Let $\mathrm{Fix}_K(\sigma) = \mathrm{Fix}(\sigma) \cap \mathrm{Max}_K(R)$.

Then $\mathfrak{m}_\alpha \in \mathrm{Fix}_K(\sigma) \iff (\underline{x} - \underline{\alpha}) = (\underline{\sigma(x)} - \alpha) \iff (\underline{x} - \underline{\sigma(x)}) \subseteq \mathfrak{m}_\alpha$,
so $\mathrm{Fix}_K(\sigma) = V(x_1 - \sigma(x_1), \ldots, x_n - \sigma(x_n)) \cap \mathrm{Max}_K(R)$,
which is a linear subspace of $\mathrm{Max}_K(R) \simeq K^n$ of codimension equal to the rank of $(\mathrm{id} - \sigma)$ as a linear transformation on $[R]_1$.

<u>Def</u>: We say that $\sigma \in G \setminus \{e\}$ is a <u>pseudoreflection</u> if $\mathrm{Fix}_K(\sigma) \subseteq \mathrm{Max}_K(R) \simeq K^n$ has codimension one.