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Examples of rings of differential operators
 we now want to compute to action rings of differential operators on rings other than poly.

Rings. Exercise: Let R be a local ring that is module-finite over a coefficient field K . Show that D_{RK} is $\text{Hom}_K(R, R)$.

~~Exercise: Let R be a field, and R be \mathbb{Z} -finite over \mathbb{Z} . Then $D_{RK} \cong \text{Hom}_{\mathbb{Z}}(R, R)$.~~

We will develop a few methods/tricks to deal with these computations. First, we note:

Prop: Let R be a graded A -algebra, with $A \subseteq [R]_0$, and grading group G . Then D_{RA} is a G -graded algebra.

pf: We observe that $R \otimes A R$ admits a G -grading by setting $[R \otimes A R]_G = \bigoplus_{a+b=G} [R]_a \otimes_a [R]_b$. The ideal

D_{RA} is homogeneous, as it is generated by homogeneous elements $\{r \otimes 1 - 1 \otimes r \mid r \text{ homog.}\}$. Thus, D_{RA} is

graded, and this is compatible with the grading on R , so D_{RA} is a graded R -module. Then

$D_{RA}^i \cong \text{Hom}_R(P_{RA}^i, R)$ is a graded R -module for each i .

We claim that $S \in D_{RA}^i$ is homogeneous at degree $t \iff S([R]_a) \subseteq [R]_{a+t}$ for each a .

Indeed, write $S = \varphi \circ d_R$, $\varphi: P_{RA}^i \rightarrow R$ of deg. t .

Then if $r \in [R]_a$, $S(r) = \varphi(1 \otimes r)$ has deg $a+t$, since $\deg(1 \otimes r) = a$. The converse is the same.

It is clear that this grading is compatible with multiplication.

Now, we recall that any diff'l op. extends uniquely to a localization. Suppose R is a ring, $W \in R$ such that w is no zero divisor, so $R_{(w)}$ is a ring. Then

~~δ~~ a map in $D_{\text{Diff}(R)}$ is an extension of an operator in $D_{\text{Diff}} \iff S(R) \subseteq R$. Put together, $D_{\text{Diff}}^{\delta} \underset{\text{res.}}{\leftarrow} \{S \in D_{\text{Diff}(R)} \mid S(R) \subseteq R\}$.

The cuspidal plane curve.

Let K be a field of char. 0, and $R = K[x^2, x^3] \subseteq S = K[x]$.

Let $T = K[x, x^{-1}]$. Observe that $Rx^2 = Sx = T$.

Thus, $D_{T/K}^i = K\{1, x^2, x^{-2}, \dots\} \cdot D_{S/K}^i = K \cdot \{x^n \partial^m \mid n \in \mathbb{Z}, m \leq i\}$.

Moreover, $D_{T/K}$, $D_{S/K}$ are graded, and

$$[D_{T/K}]_t = K \cdot \{x^{m+t} \partial^m \mid m \geq 0\}.$$

By previous discussion,

$$D_{T/K} = \{S \in D_{T/K} \mid S(R) \subseteq R\}, \text{ and}$$

this preserves order and grading.

Since $x^2 S \subseteq R$, we have $x^2 D_{S/K} \subseteq D_{T/K}$: i.e.,

$$K \cdot \{x^n \partial^m \mid n \geq 2, m \geq 0\} \subseteq D_{T/K}. [D_{T/K}]_t \supseteq x^2 [D_{S/K}]_{t-2}.$$

$$x^2 [D_{S/K}]_{t-2} = K \cdot \{x^{m+t} \partial^m \mid m \geq 0\}.$$

We will compute $[D_{T/K}]_t$ for various degrees t .

$t \geq 2$) In this case, $[D_{T/K}]_t = x^2 [D_{S/K}]_{t-2}$,

$$\text{so } [D_{T/K}]_t = [D_{S/K}]_{t-2}.$$

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$$t=1) [D_{TIK}]_1 = k\{\bar{x}\} + \bar{x}[D_{SIK}]_{-1}.$$

No ^{nonzero} element of $k\{\bar{x}\}$ stabilizes R :

$$(\lambda\bar{x})\{1\} = \lambda\bar{x} \notin R.$$

$$t=0) [D_{TIK}]_0 = k\{\bar{x}, \bar{x}\partial\} + \bar{x}^2[D_{SIK}]_{-2}$$

Both \bar{x} and $\bar{x}\partial$ stabilize R ...

$$t=-1) [D_{TIK}]_{-1} = k\{\bar{x}^{-1}, \partial, \bar{x}\partial^2\} + \bar{x}^2[D_{SIK}]_{-3}.$$

Note that these operators send $[R]_{-3}$ into R .

So, we check $1, \bar{x}^2 \dots$ find that

it is spanned by $\partial - \bar{x}\partial^2$.

$$t=-2) \text{ similarly, } [D_{TIK}]_{-2} = k\{\bar{x}^{-2}\partial - \partial^2\} + \bar{x}^2[D_{SIK}]_{-4}.$$

$$t < -2) [D_{TIK}]_t = k\{\bar{x}^{-t}, \bar{x}^{t+1}\partial, \dots, \bar{x}\partial^{-t+1}\} + \bar{x}^2[D_{SIK}]_{t-2}.$$

It suffices to check these ops. send

$1, \bar{x}, \bar{x}^2, \dots, \bar{x}^{-t+1}$ into R . One gets a system

of linear equations, and one can check that
for each such t , there is a 2-dim k -vs. V_t
such that $[D_{TIK}]_t = V_t + \bar{x}^2[D_{SIK}]_{t-2}$.

In particular, one finds $3\bar{x}^2\partial - 3\bar{x}^2\partial^2 + \partial^3 \in [D_{TIK}]_{-3}$.

One can verify that

$$\begin{aligned} D_{RIK} &= \bar{R}\langle \bar{x}\partial, \bar{x}^2\partial, \bar{x}\partial^2 - \partial, \partial^2 - 2\bar{x}^2\partial, \partial^3 - 3\bar{x}^2\partial^2 + 3\bar{x}^2\partial \rangle. \\ &\subseteq D_{TIK}, \end{aligned}$$

by checking this generates D_{RIK} in positive degrees,
and that R is a subring quotiented by $\overline{[D_{SIK}]}$. $\bar{x}^2 D_{SIK}$ is
generates $\bar{x}^2 D_{SIK}$, and

2-dim in each degree.

$$e^{-\int_{D_{11K}}^x} \left[D_{11K} + x \int_{D_{11K}}^x \right] = k(x) \quad (1=1)$$

the solution of the differential equation
 $y' + p(x)y = q(x)$

$$e^{-\int_{D_{11K}}^x} \left[D_{11K} + x \int_{D_{11K}}^x \right] = k(x) \quad (0=1)$$

both sides multiply by

$$e^{\int_{D_{11K}}^x} \left[D_{11K} + x \int_{D_{11K}}^x \right] = k(x) \quad (1=1)$$

both sides divide by $k(x)$

then we get the general solution

$$k(x) - C_1 e^{\int_{D_{11K}}^x}$$

$$e^{-\int_{D_{11K}}^x} \left[D_{11K} + x \int_{D_{11K}}^x \right] = k(x) \quad (C_1 = 1)$$

$$e^{-\int_{D_{11K}}^x} \left[D_{11K} + \{ \dots \} \right] = k(x) \quad (C_1 = 1)$$

it suffices to take the limit

as $x \rightarrow \infty$ gives a unique solution C_1 .

then we can write the general solution of linear differential equations

$$e^{-\int_{D_{11K}}^x} \left[D_{11K} + C_1 \right] = k(x) \quad \text{take limit}$$

$$e^{-\int_{D_{11K}}^x} \left[D_{11K} + C_1 \right] = k(x) \quad \text{and we multiply it}$$

we can now find

$$\langle C_1 x^3 + C_1 x^2 - C_1 x - C_1, C_1 x^3, C_1 x, C_1 \rangle = k(x)$$

$$e^{-\int_{D_{11K}}^x}$$

as specified this formular is in positive degrees

and this formular is a consequence of the first fundamental theorem of calculus

function & derivative