Examples of rings of differential operators

We now want to compute to actual rings of differential operators on rings other than poly. rings.

Exercise: Let R be a local ring that is module-finite over a coefficient field K. Show that $D_{R|K}$ is $\mathrm{Hom}_K(R,R)$.

~~Example: Let K be a field, and R be a finite algebra K-algebra. Then $D_{R|K} = \mathrm{Hom}_K(R,R)$.~~

We will develop a few methods/tricks to deal with these computations. First, we note:

Prop: Let R be a graded A-algebra, with $A \subseteq [R]_0$, and grading group G. Then $D_{R|A}$ is a G-graded A-algebra.

prf: We observe that $R \otimes_A R$ admits a G-grading by setting $[R \otimes_A R]_c = \bigoplus_{a+b=c} [R]_a \otimes_A [R]_b$. The ideal $\Delta_{R|A}$ is homogeneous, as it is generated by homog. elements $\{r \otimes 1 - 1 \otimes r \mid r \text{ homog.}\}$. Thus, $P^i_{R|A}$ is graded, and this is compatible with the grading on R, so it is a graded R-module. Then $D^i_{R|A} \simeq \mathrm{Hom}_R(P^i_{R|A}, R)$ is a graded R-module for each i.

We claim that $\delta \in D^i_{R|A}$ is homogeneous of degree $t \iff \delta([R]_a) \subseteq [R]_{a+t}$ for each a.

Indeed, write $\delta = \varphi \circ d_R$, $\varphi : P^i_{R|A} \to R$ of deg. t. Then if $r \in [R]_a$, $\delta(r) = \varphi(1 \otimes r)$ has deg $a+t$, since $\deg(1 \otimes r) = a$. The converse is the same.

It is then clear that this grading is compatible with multiplication.

Now, we recall that any diff'l op. extends uniquely to a localization. Suppose R is a ring (A-alg), $W \subseteq R$ mult ~~set~~ w/no zerodivisors, so $R \subseteq W^{-1}R$. Then a map δ in $D^i_{W^{-1}R|A}$ is an extension of an operator in $D^i_{R|A}$ $\iff$ $\delta(R) \subseteq R$. Put together,

$$D^i_{R|A} \underset{\text{res.}}{\xleftarrow{\sim}} \{\delta \in D^i_{W^{-1}R|A} \mid \delta(R) \subseteq R\}.$$

~~Fhe~~ The cuspidal plane cubic.

Let k be a field of char 0, and $R = k[x^2, x^3] \subseteq S = k[x]$.

Let $T = k[x, x^{-1}]$. Observe that $R_{x^2} = S_x = T$.

Thus, $D^i_{T|k} = k\{1, \bar{x}^{-1}, \bar{x}^{-2}, \dots\} \cdot D^i_{S|k} = k \cdot \{\bar{x}^n \partial^m \mid n \in \mathbb{Z}, m \le i\}$.

Moreover, $D_{R|k}$, $D_{T|k}$ are graded, and

$$[D_{T|k}]_t = k \cdot \{\bar{x}^{m+t}\partial^m \mid m \ge 0\}.$$

By previous discussion,

$$D_{R|k} = \{\delta \in D_{T|k} \mid \delta(R) \subseteq R\},$$

and this preserves order and grading.

Since $x^2 S \subseteq R$, we have $\bar{x}^2 D_{S|k} \subseteq D_{R|k}$: i.e.,

$k \cdot \{\bar{x}^n \partial^m \mid n \ge 2, m \ge 0\} \subseteq D_{R|k}$. $[D_{R|k}]_t \supseteq \bar{x}^2 [D_{S|k}]_{t-2}$.

$\bar{x}^2 [D_{S|k}]_{t-2} = k \cdot \{\bar{x}^{m+t}\partial^m \mid m \ge 0,\ m+t \ge 2\}$.

We will compute $[D_{R|k}]_t$ for various degrees t.

$t \ge 2$) In this case, $[D_{T|k}]_t = \bar{x}^2 [D_{S|k}]_{t-2}$,

so $[D_{R|k}]_t = [D_{T|k}]_t$.

$t=1$) $[D_{T|K}]_1 = k\cdot\{\bar{x}\} + \bar{x}^2[D_{S|K}]_{-1}$.

No nonzero element of $k\{\bar{x}\}$ stabilizes R:

$(\lambda\bar{x})(1) = \lambda x \notin R$.

$t=0$) $[D_{T|K}]_0 = k\{I, \bar{x}\partial\} + \bar{x}^2[D_{S|K}]_{-2}$

Both I and $\bar{x}\partial$ stabilize R ... --

$t=-1$) $[D_{T|K}]_{-1} = k\{\bar{x}^{-1}, \partial, \bar{x}\partial^2\} + \bar{x}^2[D_{S|K}]_{-3}$.

Note that these operators send $[R]_{\geq 3}$ into R.

So, we check $1, x^2 \ldots$ find that it is spanned by $\partial - \bar{x}\partial^2$.

$t=-2$) similarly, $[D_{R|K}]_{-2} = k\{2\bar{x}^{-1}\partial - \partial^2\} + \bar{x}^2[D_{S|K}]_{-4}$.

$t<-2$) $[D_{T|K}]_t = k\{\bar{x}^{t}, \bar{x}^{t+1}\partial, \ldots, \bar{x}\partial^{-t+1}\} + \bar{x}^2[D_{S|K}]_{t-2}$.

It suffices to check these ops. send $1, x^2, x^3, \ldots, x^{-t+1}$ into R. One gets a system of linear equations, and one can check that for each such t, there is a 2-dim k-v.s. V_t such that $[D_{R|K}]_t = V_t + \bar{x}^2[D_{S|K}]_{t-2}$.

In particular, one finds $3\bar{x}^{-2}\partial - 3\bar{x}^{-1}\partial^2 + \partial^3 \in [D_{R|K}]_{-3}$.

One can verify that

$$D_{R|K} = \bar{R}\langle \bar{x}\partial, \bar{x}^2\partial, \bar{x}\partial^2 - \partial, \partial^2 - 2\bar{x}^{-1}\partial, \partial^3 - 3\bar{x}^{-1}\partial^2 + 3\bar{x}^{-2}\partial\rangle.$$
$$\subseteq D_{T|K},$$

by checking this generates $D_{R|K}$ in positive degrees, and that this subring generates $\bar{x}^2 D_{S|K}$, and quotiented by ~~$[D_{S|K}]$~~ $\bar{x}^2 D_{S|K}$ is 2-dim in each degree.