[redacted]

Prop: If K is perfect, and $(R, \mathfrak{m}, k)$ is a [redacted] eft$/K$, then $\exists$ quasicoefficient field L for R containing K.

pf: $K \subseteq k$ is a f.g. field extension: it is generated by the images of the gens of R up to localization.
Pick $x_1, \dots, x_t \in R$ s.t. the images $\{\tilde{x}_i\}$ in $R/\mathfrak{m} \cong k$ ~~for~~ are alg. indept. and $K(\tilde{x}_1, \dots, \tilde{x}_t) \subseteq k$ is fin. sep'l, by MacLane.
We claim that [redacted] $\{x_1, \dots, x_t\}$ are alg. indep't over K; otherwise, a alg. relation on these would give a nonzero alg. relation on the $\tilde{x}_i$'s in k.
We then observe that any nonzero polynomial in the x_i's is a unit in R; else we get a relation mod $\mathfrak{m}$ (i.e. on the $\tilde{x}$'s in k). Thus $K(x_1, \dots, x_t) = L$ is a purely transcendental$/K$ subfield of R, and the image of L to k is finite separable.
That is, L is a quasicoefficient field.

[redacted].

Ex: $\mathbb{R}[x]_{(x^2+1)}$ has no ~~quasi~~ coefficient field, since there is no solution to $f^2+1=0$ in $\mathbb{R}(x) \cong \mathbb{R}[x]_{(x^2+1)}$. $\mathbb{R}$ is a quasicoefficient field though.

Ex: $\mathbb{F}_p(t)[x]_{(x^p-t)}$ has no quasicoefficient field, even, since there is no solution to $f^p=t$ in $\mathbb{F}_p(t,x) \cong \mathbb{F}_p(t)[x]_{(x^p-t)}$.

Prop: Let $(R,\mathfrak{m},k)$ be a local ring with quasicoefficient field L. Then $k\otimes_R P^n_{R|L} \cong R/\mathfrak{m}^{n+1}$ as $R\otimes_L R$-modules, where the left action on $R/\mathfrak{m}$ is described below, and right action is the usual R-action.

Prf: $k\otimes_R P^n_{R|L} \cong k\otimes_R \dfrac{R\otimes_L R}{\Delta^{n+1}_{R|L}} \cong \dfrac{k\otimes_L R}{\overline{\Delta}^{n+1}_{R|L}}$, where $\overline{\Delta}_{R|L}$ is the image of $\Delta_{R|L}$ modulo $\mathfrak{m}\otimes_L R$.

We can write $k = L(\lambda) = L[T]/(f(T))$ by primitive element theorem, where f is the min poly of f over L. Then $f'(\lambda)$ is nonzero in k by separability. If $\gamma\in R$ has image $\lambda\in R/\mathfrak{m}=k$, then $f(\gamma)\in\mathfrak{m}$ and $f'(\gamma)$ is a unit in R. By Hensel's Lemma, we can pick γ with $f(\gamma)\in\mathfrak{m}^{n+1}$!

We claim that $\overline{\Delta}_{R|L} = (1\otimes\mathfrak{m} + (1\otimes\gamma-\lambda\otimes 1))$. Indeed, we can write $r = m+g(\gamma)$ for $m\in\mathfrak{m}$, $g(T)\in L[T]$, and

$$1\otimes r - r\otimes 1 = (1\otimes m - m\otimes 1) + (1\otimes g(\gamma) - g(\gamma)\otimes 1)$$
$$\equiv 1\otimes m + (1\otimes g(\gamma) - g(\lambda)\otimes 1) \mod \mathfrak{m}\otimes_L R,$$

and the latter is an $R\otimes_L R$-linear combination of $1\otimes\gamma - \lambda\otimes 1$ (exercise).

Now, $k\otimes_L R \cong R[x]/(f(x))$ as L-algebras. "Left R" acts by taking $r \bmod \mathfrak{m} \in k$ as a polynomial in $L[x]/(f(x))$, and multiplying. "Right R" is the usual action.

The image of $I_{R|L}$ is $m + (\gamma - x) =: Q$.

We claim that $m^{n+1} + (\gamma - x) = Q^{n+1}$ in $R[x]/(f(x))$. "$(\supseteq)$" is clear. To see ~~this~~ equality, note that Q is the only maximal ideal containing $m^{n+1} + (\gamma - x)$, so we may localize at Q and apply NAK. But, we have

$$0 = f(x) = f((x-\gamma)+\gamma) = f(\gamma) + f'(\gamma)(x-\gamma) + H\cdot(x-\gamma)^2.$$

Thus $\cancel{x}-\gamma = m + (x-\gamma)^2 h \in m^{n+1} + (x-\gamma)^2$

Then $m^{n+2} + (\gamma - x) \subseteq Q^{n+2} + Q(m^{n+2} + (\gamma - x))$ holds, since $m^{n+1} \subseteq Q^{n+1}$ and $(\gamma - x) \subseteq Q^{n+1} + (\gamma - x)Q$.

Thus, $\left(R[x]/(f(x))\right)/Q^{n+1} \xrightarrow{\sim} \left(R[x]/(f(x))\right)/(m^{n+1} + (\gamma - x))$
$\simeq R/m^{n+1}$, since $f(\gamma) \in m^{n+2}$,

as required. The left action of R is by evaluating mod m and writing as $g(\lambda)$, then acting by $g(\gamma)$.

Prop: Let (R, m, k) be local with quasicoefficient field L. Then $D^n_{R|L}(R,k) \simeq \operatorname{Hom}_k(R/m^{n+1}, k)$ (by the left R-mod structure above). In particular, if $f \notin m^{n+1}$, $\exists\, \delta \in D^n_{R|L}(R,k)$ such that $\delta(f) = 1$.

Pf: We have $D^n_{R|L}(R,k) \simeq \operatorname{Hom}_R(P^n_{R|L}, k) \simeq \operatorname{Hom}_k(k \otimes_R P^n_{R|L}, k)$
$\simeq \operatorname{Hom}_k(R/m^{n+1}, k)$.

The second claim comes from observing that a nonzero element in R/m^{n+1} is part of a k-basis. $\boxtimes$

<u>Theorem</u>: Let k be a perfect field, and R be ess. of finite type over k, $P \subseteq R$ prime. Then

$$P^{(n)} = (0 :_R D^{n-1}_{R|k}(R, R/P)).$$

<u>pf</u>: We already have $P^{(n)} \subseteq (0 :_R D^{n-1}_{R|k}(R, R/P))$.
It suffices to show $P^{(n)}R_P = P^{(n)}R_P \supseteq (0 :_R D^{n-1}_{R|k}(R, R/P))R_P$.
We have RHS $= (0 :_{R_P} D^{n-1}_{R_P|k}(R_P, R_P/PR_P))$.
Write $(R_P, PR_P, R_P/PR_P) = (S, \mathfrak{m}, \kappa)$: S is ess. fin. type over perfect k. There is a coeff. field L for S. Now, $(0 :_S D^{n-1}_{S|k}(S, \kappa)) \subseteq (0 :_S D^{n-1}_{S|L}(S, \kappa))$, since $k \subseteq L$, ~~But~~ $D^{n-1}_{S|L}(S, \kappa) \subseteq D^{n-1}_{S|k}(S, \kappa)$.
But, if $f \notin \mathfrak{m}^n$, then the image of f is nonzero in $S/\mathfrak{m}^n$, so there is a map in $\mathrm{Hom}_L(S/\mathfrak{m}^n, \kappa) \cong D^{n-1}_{S|L}(S, \kappa)$ taking f to something nonzero, so $f \notin (0 :_S D^{n-1}_{S|L}(S, \kappa))$, as required. $\boxed{}$

<u>Theorem</u>: Let k be a perfect field, and R be (a localization of) a poly ring over k, $P \subseteq R$ prime. Then

$$P^{(n)} = (P :_R D^{n-1}_{R|k}).$$

<u>pf</u>: We just need to show that $(P :_R D^{n-1}_{R|k}) = (0 :_R D^{n-1}_{R|k}(R, R/P))$ in this case.

From the short exact sequence

$$0 \to P \to R \to R/P \to 0,$$ we get

$$0 \to Hom_R(P^n_{R|k}, P) \longrightarrow Hom_R(P^n_{R|k}, R) \to Hom_R(P^n_{R|k}, R/p) \to 0$$

since $P^n_{R|k}$ is a free module (crucial). Thus

$$D^n_{R|k} \longrightarrow D^n_{R|k}(R, R/p)$$

$$\delta \longmapsto \pi \circ \delta$$, where $\pi : R \to R/p$ projection

is surjective. That is, every op. to R/p comes from op. $R \to R$.

Thus, if $r \in (P :_R D^{n-1}_{R|k})$, ~~then~~ and

$\beta \in D^{n-1}_{R|k}(R, R/p)$, write $\beta = \pi \circ \delta$, $\delta \in D^{n-1}_{R|k}$,

so $\beta(r) = \pi(\delta(r)) = 0$. Thus $(P :_R D^{n-1}_{R|k}) \subseteq (0 :_R D^{n-1}_{R|k}(R, R/p))$.

Conversely, if $r \in (0 :_R D^{n-1}_{R|k}(R, R/p))$, and $\delta \in D^{n-1}_{R|k}$,

then $(\pi \circ \delta) \in D^{n-1}_{R|k}(R, R/p)$, so $(\pi \circ \delta)(r) = 0$,

which means $\delta(r) \in P$. Thus, $(0 :_R D^{n-1}_{R|k}(R, R/p)) \subseteq (P :_R D^{n-1}_{R|k})$. ∎