Zariski-Nagata theorems

Our next application is to symbolic powers. We want to prove a theorem ~~of~~ Zariski-Nagata & a more general version by Cid-Ruiz building on work of Brumfiel.

We will use colon notation
$$(N :_R T) = \{ r \in R \mid t(r) \in N \ \forall t \in T \}$$
for a collection of differential operators T from $R \to M$ and some submodule $N \subseteq M$.

Our goal is the following theorem:

Thm: Let k be a perfect field, R be an algebra ess. of finite type over k, and $P \subseteq R$ prime. Then

1) $P^{(n)} = (0 :_R D^{n-1}_{R|k}(R, R/P))$, and
2) If R (is a localization of) a poly. ring, then $P^{(n)} = (P :_R D^{n-1}_{R|k})$.

<u>First containment</u>

This will require a bit of preparation.

First we study the ideals specified on the RHS's.

<u>Prop</u>: For each n, and any ideal I, $(0 :_R D^n_{R|A}(R, R/I))$ and $(I :_R D^n_{R|A})$ are ideals.

<u>prf:</u> By induction on n.

For $n=0$, $(0 :_R D^0_{R|A}(R, R/I)) = (I :_R D^0_{R|A}) = I$.

<u>Ind. step:</u> Additivity is clear, since operators are additive.

If $r \in R$ with $D^n_{R|A}(R, R/I) \cdot r = 0$ and $s \in R$ general,

$D^n_{R|A}(R, R/I) \circ s \subseteq D^n_{R|A}(R, R/I)$, $\Rightarrow$ OK.

Likewise for $D^n_{R|A} \cdot r \in I$.

Prop: We have $I(0 :_R D^i_{R|A}(R, R/I)) \subseteq (0 :_R D^{i+1}_{R|A}(R, R/I))$
and $I(I :_R D^i_{R|A}) \subseteq (I :_R D^{i+1}_{R|A})$.

pf: Let $D^i_{R|A}(R, R/I) \cdot r = 0$ and $f \in I$.
If $\delta \in D^{i+1}_{R|A}(R, R/I)$, then $\delta(fr) = \delta \bar{f}(r) = \bar{f}\delta(r) + [\delta, f](r)$
$= \underbrace{f\delta(r)}_{0} + \underbrace{0}_{IH} = 0$.

Similarly for other.

Prop: If P is prime, then $(0 :_R D^i_{R|A}(R, R/P))$
and $(P :_R D^i_{R|A})$ are P-primary.

pf: Induce to show $r \notin P$, $ra \in I \Rightarrow a \in I$ for these ideals.
If $D^{i+1}_{R|A}(R, R/P) \cdot (ra) = 0$, $r \notin P$, take $\delta \in D^{i+1}_{R|A}(R, R/P)$.
Then $0 = \delta(ra) = (\delta \bar{r})(a) = (\bar{r}\delta)(a) + [\delta, \bar{r}](a)$
$= r\,\delta(a) + [\delta, \bar{r}](a)$.
By IH, $[\delta, \bar{r}](a) \subseteq D^i_{R|A}(R, R/P) \cdot a = 0$, and
$r\,\delta(a) = 0 \Rightarrow \delta(a) = 0$.
Similarly for other. ☒

This gives the containment $P^{(n)} \subseteq (0 :_R D^{n-1}_{R|A}(R, R/P))$,
since $P^n \subseteq (0 :_R D^{n-1}_{R|A}(R, R/P))$ by above above,
and $P^{(n)}$ is smallest P-primary ideal containing P^n.

If R is e.f.t. $/k$, $P \subseteq R$ prime, then

<u>Prop</u>: $(0:_R D^{n-1}_{R|k}(R, R/p))_p = (0:_{R_p} D^{n-1}_{R_p|k}(R_p, R_p/pR_p))$.

<u>prf</u>: Can write an element of LHS as r/w with $D^{n-1}_{R|k}(R, R/p) \cdot r = 0$, $w \notin P$. If $\delta \in D^{n-1}_{R_p|k}(R_p, R_p/pR_p)$, write $\delta = \frac{1}{v}\alpha$, $v \notin P$, and α extension of an op. in $D^{n-1}_{R|k}(R, R/p)$. Then $\delta(\frac{r}{w}) = \frac{1}{vw}\left(\alpha(r) + \frac{\alpha^{(1)}(r)}{w} + \frac{\alpha^{(2)}(r)}{w^2} + \cdots + \frac{\alpha^{(n-1)}(r)}{w^{n-1}}\right) = 0$. $(\alpha^{(i)} \in D^{n-1-i}_{R|k}(R, R/p))$, so $(\subseteq)$ holds. If $D^{n-1}_{R_p|k}(R_p, R_p/pR_p) \cdot \frac{r}{w} = 0$ $(w \notin P)$, $D^{n-1}_{R_p|k}(R_p, R_p/pR_p) \cdot r = 0$, and $D^{n-1}_{R|k}(R, R/p) \cdot r = 0$ (since every op. extends to the former type), so $(\supseteq)$ too.

Thus, it only remains for (i) to show that if $(R, \mathfrak{m}, k)$ is local, e.f.t. over K perfect, then $(0:_R D^{n-1}_{R|K}(R, k)) = \mathfrak{m}^n$.

<u>Separable field extensions & quasicoefficient fields.</u>

<u>Thm</u> (MacLane): Let $K \subseteq L$ be ~~an~~ fin. gen. extension of fields, with K perfect. Then $\exists x_1, \ldots, x_t \in L$ s.t.

$K \subseteq K(x_1, \ldots, x_t)$ purely transcendental and

$K(x_1, \ldots, x_t) \subseteq L$ finite separable.

<u>prf</u>: If char $K = 0$, no problem: any tr. basis works. (Note that $L/K(x_1, \ldots, x_t)$ alg $\Rightarrow$ finite, since fin. gen.)

In char $p > 0$, let $F^{sep} = \{\ell \in L \mid \ell$ sep'l alg over $F\}$ for $F \subseteq L$ subfield. Pick $x_1, \ldots, x_t \in L$ so that $[L : K(x_1, \ldots, x_t)^{sep}]$ is minimal among all tr bases $\{x_1, \ldots, x_t\}$ (again note this is finite).

Suppose $\exists y \in L$ inseparable over $K(x_1, \ldots, x_t)$, and let $F(Z)$ be its min poly. By taking coeffs in coprime terms and clearing denominators, we get a poly

Note all exponents are multiples of p.

$H(Z) \in K[x_1, \ldots, x_t, Z]$ s.t. $H(y) = 0$, the coeffs in $K[\underline{x}]$ are relatively prime, and H is irred in $K(\underline{x})[Z]$; so H irred by Gauss' Lemma.

Now, not every exponent of each x_j is a mult. of p, else H is a p^{th} power (since k perfect), so wlog x_n occurs somewhere without a mult. of p exponent. Thus, x_n is sep'l alg over $K[x_1, \ldots, x_{n-1}, y]$, and $x_1, \ldots, x_{n-1}, y$ is a tr. basis for L/K. Then $x_n, y \in K(x_1, \ldots, x_{n-1}, y)^{sep} \supsetneq K(x_1, \ldots, x_n)^{sep}$, contradicting the choice of $x_1, \ldots, x_n$. $\boxed{4}$

Def: If (R, m, k) is a local ~~k-algebra~~ ring, we say L is a *quasicoefficient field* for R if $L \in R$ and $L \subseteq R \twoheadrightarrow k$ is finite separable.