Ex: Let A be a ring, and $R = A[x_1, \dots, x_n]$, poly ring over A. We can write

$$R \otimes_A R \simeq A[x_1, \dots, x_n, y_1, \dots, y_n]. \quad \begin{pmatrix} x_i = x_i \otimes 1 \\ y_i = 1 \otimes x_i \end{pmatrix}$$

~~Then~~ / We view $R \otimes_A R$ as an R-algebra by the left-inclusion map [and we identify elements with their image in an algebra like usual.].

Then

$$d_R(x_i) = y_i \quad \text{for each } i, \text{ so}$$

$$d_R(f(x)) = f(y).$$

The ideal $\Delta_{R|A} = (y_1 - x_1, \dots, y_n - x_n)$.

Let's rewrite $z_i = y_i - x_i$, so

$$R \otimes_A R \simeq A[x_1, \dots, x_n, z_1, \dots, z_n] \text{ and}$$

$\Delta_{R|A} = (z_1, \ldots, z_n)$. Now, $d_R(x_i) = x_i + z_i$, so
$d_R(f(\underline{x})) = f(\underline{x}+\underline{z})$.

We compute $P^i_{R|A} \simeq A[\underline{x},\underline{z}]/(\underline{z})^{i+1}$

$$\underset{R\text{-modules}}{\simeq} \bigoplus_{\alpha_1+\cdots+\alpha_n \le i} R\, z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

This is a free module on the generators $\{z^\alpha \mid |\alpha| \le i\}$.
Let $\{(z^\alpha)^* \mid |\alpha| \le i\}$ be the dual basis:
i.e., $(z^\alpha)^*$ returns the z^α-component in the unique expression of an element as a sum as above.
Thus $\mathrm{Hom}_R(P^i_{R|A}, R) \simeq \bigoplus_{|\alpha| \le i} R (z^\alpha)^*$.

We now want to identify $(z^\alpha)^* \circ d_R$ as differential operators.

Def: Let $\partial^{(\alpha)}: R \to R$ $(R = A[\underline{x}])$ be the map A-linear on R defined by

$$\partial^{(\alpha)}(x_1^{\beta_1} \cdots x_n^{\beta_n}) = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n} x_1^{\beta_1-\alpha_1} \cdots x_n^{\beta_n-\alpha_n}.$$

If $\mathbb{Q} \subseteq A$, we can identify $\partial^{(\alpha)}$ with $\frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n} \frac{1}{\alpha_1! \cdots \alpha_n!}$,
as $\binom{\beta_i}{\alpha_i} = \frac{\beta_i(\beta_i-1)\cdots(\beta_i-\alpha_i+1)}{\alpha_i(\alpha_i-1)\cdots 1}$.
In general, $1/(\alpha_1! \cdots \alpha_n!)$ is not necessarily meaningful in A.

Lem: (Taylor expansions): In $R=A[\underline{x}]$, we have $f(\underline{x}+\underline{z}) = \sum_{\alpha} (\partial^{(\alpha)} f)(\underline{x})\cdot \underline{z}^{\alpha}$.

prf: It suffices to check for monomials, in which case this is basically the binomial theorem.

Theorem: Let $R=A[x_1,\dots,x_n]$ be a poly ring over another ring. Then

$$D^i_{R|A} = \bigoplus_{|\alpha|\le i} R\,\partial^{(\alpha)}.$$

prf: We computed that $\mathrm{Hom}_R(P^i_{R|A}, R)$ is freely gen. by $(\underline{z}^{\alpha})^*$, so $D^i_{R|A}$ is freely gen. by $\partial^{(\alpha)}$ (using the Taylor expansion). $\boxed{}$.

Exer: Show that $D_{A[x_1,\dots,x_n]|A}$ is f.g. A-alg $\iff \mathbb{Q}\subseteq A$.

We also note the following:

Prop: Let $R=A[\underline{x}]$ be a poly ring. Then $P^i_{R|A}$ is a free R-module.

Our next goal is to give some sort ~~of~~ of description of differential operators on finitely-generated algebras. The key is the following:

Prop: Let $S=A[x_1,\dots,x_n]$ poly, $I\subseteq S$ an ideal, and $R=S/I$. Then $\forall \delta\in D^i_R, \exists \tilde{\delta}\in D^i_S$ s.t.

$$\begin{array}{ccc} S & \xrightarrow{\tilde{\delta}} & S \\ \downarrow & & \downarrow \\ R & \xrightarrow{\delta} & R \end{array}$$

commutes.

<u>prf</u>: Note ~~that~~ $R \otimes_A R \simeq S/I \otimes_A S/I$ is quotient of $S \otimes_A S$, and $\Delta_{S|A}$ maps to $\Delta_{R|A}$ under this quotient. Thus, we have a surjection $P^i_{S|A} \twoheadrightarrow P^i_{R|A}$; moreover

$$\begin{tikzcd} S \arrow[r, "d_S"] \arrow[d] & P^i_{S|A} \arrow[d] \\ R \arrow[r, "d_R"] & P^i_{R|A} \end{tikzcd}$$

commutes.

Given δ, write $\delta = \varphi \circ d_R$, so we have

$$\begin{tikzcd} S \arrow[r, "d_S"] \arrow[d] & P^i_{S|A} \arrow[r, dashed, "\tilde{\varphi}"] \arrow[d] & S \arrow[d] \\ R \arrow[r, "d_R"] & P^i_{R|A} \arrow[r, "\varphi"] & R \end{tikzcd}$$

Then $P^i_{S|A} \to R$ is S-linear, and by freeness ($P^i_{S|A}$ is projective), there is a (S-linear) lift $\tilde{\varphi}$ making the diagram commute. Then $\tilde{\varphi} \circ d_S = \tilde{\delta}$ works.

<u>Thm</u>: Let $S = A[\underline{X}]$ poly ring, $I \subseteq S$ ideal, $R = S/I$. Then
$$D^i_{R|A} \cong \frac{\{\delta \in D^i_{S|A} \mid \delta(I) \subseteq I\}}{I D^i_{S|A}}.$$

<u>pf</u>: By last proposition, every op. in $D^i_{R|A}$ lifts to an op. in $D^i_{S|A}$. ~~Can~~ op. $\delta \in D^i_{S|A}$ ~~is~~ the lift of some element in $D^i_{R|A}$ $\iff \delta(I) \subseteq I$. $(\Leftrightarrow \delta(s+I) \subseteq \delta(s)+I)$

Thus, $$D^i_{R|A} \cong \frac{\{\delta \in D^i_{S|A} \mid \delta(I) \subseteq I\}}{\{\delta \in D^i_{S|A} \mid \delta(S) \subseteq I\}}.$$

Exercise to check $\delta(S) \subseteq I \iff \delta \in I D^i_{S|A}$. ∎

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Now we use principal parts ~~to~~ investigate the behavior ~~of~~ differential operators under localization.

Prop: Let $A \to R$ commut., and $W \subseteq R$ mult. closed. Then
$$W^{-1} P^i_{R|A} \simeq P^i_{W^{-1}R|A} \simeq P^i_{W^{-1}R|(W\cap A)^{-1}A}.$$

pf: $W^{-1}P^i_{R|A} = (W\otimes 1)^{-1}\left(\frac{R\otimes_A R}{\Delta^{i+1}_{R|A}}\right)$, and
$$P^i_{W^{-1}R|A} = \frac{W^{-1}R \otimes_A W^{-1}R}{\Delta^{i+1}_{W^{-1}R|A}} \simeq (W\otimes 1)^{-1}(1\otimes W)^{-1}\left(\frac{R\otimes_A R}{\Delta^{i+1}_{R|A}}\right),$$
so we need to show that any element of the form $1\otimes w$, $w\in W$ is a unit in $(W\otimes 1)^{-1}\left(\frac{R\otimes_A R}{\Delta^{i+1}_{R|A}}\right)$. But $1\otimes w = w\otimes 1 + (1\otimes w - w\otimes 1)$, a unit plus a nilpotent, which is a unit (since for any prime ideal I, nilpotent $\in I$, unit $\notin I$).

This justifies the first isomorphism.

For the second, the map $W^{-1}R\otimes_A W^{-1}R \to W^{-1}R \otimes_{(W\cap A)^{-1}A} W^{-1}R$ sending $a/w \otimes b/v \mapsto a/w \otimes b/v$ is well-defined (since there is an A-bilinear map $W^{-1}R\times W^{-1}R \to$ ——), and the inverse is well-defined too. This induces the isomorphism on the respective quotients.

(since $\frac{a}{w}\frac{\alpha}{\mu}\otimes\frac{b}{v} = \frac{a(\alpha\mu)}{\mu w}\otimes\frac{b}{\mu v}$ in $W^{-1}R\otimes_A W^{-1}R$) $= \frac{a}{\mu w}\otimes\frac{(\alpha\mu)b}{\mu v} = \frac{a}{w}\otimes\frac{\alpha}{\mu}\frac{b}{v}$

<u>Prop:</u> Let R be a localization of a finitely generated A-algebra. Then $P^i_{R|A}$ is a finitely generated R-module.

<u>prf:</u> By the last proposition, it suffices to deal with the case R is finitely generated. Write $R = S/I$, for S a f.g. poly ring over A. We then have

$$P^i_{S|A} = \frac{S\otimes_A S}{\Delta^{i+1}_{S|A}} \twoheadrightarrow \underbrace{\frac{S\otimes_A S}{I\otimes 1 + 1\otimes I + \Delta^{i+1}_{S|A}}}_{\text{is}\ \ \frac{R\otimes_A S}{I\otimes I + \Delta^{i+1}_{S|A}(R\otimes_A S)}} \simeq P^i_{R|A}$$

A generating set for $P^i_{S|A}$ serves as a gen. set for $(R\otimes_A S)/\Delta^{i+1}_{S|A}(R\otimes_A S)$, which surjects onto $P^i_{R|A}$. The claim follows.

A Noeth

<u>Thm:</u> Let R be a localization of a f.g. A-algebra and M an R-mod. Then $D^{i\,(W^{-1}R,\,W^{-1}M)}_{W^{-1}R|A} \simeq W^{-1} D^{i\,(R,M)}_{R|A}$ as $W^{-1}R$-modules.

$\simeq D^{i}_{W^{-1}R|(W\cap A)^{-1}A}$

<u>Prf:</u> We have

$$D^{i\,(W^{-1}R,\,W^{-1}M)}_{W^{-1}R|A} \simeq \mathrm{Hom}_{W^{-1}R}(P^i_{W^{-1}R|A}, W^{-1}M) \cancel{\simeq W^{-1}\mathrm{Hom}_R(P}$$

$$\simeq \mathrm{Hom}_{W^{-1}R}(W^{-1}P^i_{R|A}, W^{-1}M)$$

$$\simeq W^{-1}\mathrm{Hom}_R(P^i_{R|A}, M) \quad \left(\begin{array}{l}\text{by hypothesis, since} \\ P^i_{R|A} \text{ is f.g., and } R \text{ is Noeth} \\ \quad\Rightarrow P^i_{R|A} \text{ fin. pres.}\end{array}\right)$$

$$\simeq W^{-1} D^i_{R|A}(R,M).$$

More generally, this holds whenever $P^i_{R|A}$ is finitely presented.

The other is similar.

This theorem says that in this case, every diff operator on $W^{-1}R$ is $\frac{1}{w}\circ\tilde{\delta}$ for $\tilde{\delta}$ an extension of an operator on R to $W^{-1}R$. We can make this more explicit:

Let $\delta \in D^{i}_{R|A}(R,M)$ and $w \in W$. If $\tilde{\delta}$ is an extension of δ to $D_{W^{-1}R|A}(W^{-1}R, W^{-1}M)$, we must have

$$\delta(r) = \tilde{\delta}\left(w\tfrac{r}{w}\right) = (\tilde{\delta}\,\bar{w})\left(\tfrac{r}{w}\right) = \left(\bar{w}\tilde{\delta} + [\tilde{\delta},\bar{w}]\right)\left(\tfrac{r}{w}\right)$$
$$= w\,\tilde{\delta}(r/w) + [\tilde{\delta},\bar{w}](r/w),$$

so
$$\tilde{\delta}(r/w) = \frac{\delta(r) - [\tilde{\delta},\bar{w}](r/w)}{w}$$

and $[\tilde{\delta},\bar{w}]$ has smaller order, so this inductively gives an operator. Repeating, write $\delta^{(0)} = \delta$, $\delta^{(j)} = [\delta^{(j-1)},\bar{w}] \in D^{i-j}_{R|A}(R,M)$; we get

$$\tilde{\delta}(r/w) = \sum_{j=0}^{i} \frac{\delta^{(j)}(r)}{w^{j+1}}.$$