Exer: $\mathrm{gr}(D_{R|k})$ is commutative!

Principal parts

To better understand differential operators, we want an "R-linear" way to compute them, i.e., to represent them as a functor. To do this, we'll need to represent Hom over a subring first.

Module structures on Hom

Let $A \to R$ be a homomorphism of commutative rings, and let M, N be R-modules. There is an A-module homomorphism

$$d_M : M \longrightarrow R \otimes_A M$$
$$m \longmapsto 1 \otimes m.$$

Observe that there are different R-module structures on $R \otimes_A M$:

we can act on R: $\quad r \cdot (s \otimes m) = rs \otimes m$

or act on M: $\quad r \cdot (s \otimes m) = r \otimes sm$,

and these differ in general. However, the action of A on R or on M agree:

$\alpha s \otimes m = s \otimes \alpha m$ for $\alpha \in A$ by bilinearity over A.

When we consider $R \otimes_A M$ as an R-module, it will be by the action on R.

There is an $(R \otimes_A R)$-module structure on $R \otimes_A M$:

$(a \otimes b)(r \otimes m) = ar \otimes bm$.

Note that if $\alpha, \beta \in A$, we have $\alpha a \otimes b = a \otimes \alpha b$, $\beta r \otimes m = r \otimes \beta m$, and $\alpha \beta a r \otimes bm = ar \otimes \alpha \beta bm$... the action is well-defined.

Beware: d_M is not R-linear under this structure

$\mathrm{Hom}_A(M,N)$ also admits an $(R\otimes_A R)$-mod structure.
For $\phi \in \mathrm{Hom}_A(M,N)$, $(a\otimes b)\phi = a\phi(bm)$.
we have $(\alpha a\otimes b) = (a\otimes \alpha b)$
and $\alpha a\phi(bm) = a\phi(\alpha bm)$ by A-linearity.

<u>Theorem</u>: The map
$$\mathrm{Hom}_R(R\otimes_A M, N) \xrightarrow{\Phi} \mathrm{Hom}_A(M,N)$$
$$\Psi \longmapsto \Psi \circ d_M$$
is an isomorphism of $(R\otimes_A R)$-modules
(where the action on LHS is by precomposition).
<u>pf</u>: First, we verify that this map is $(R\otimes_A R)$-linear: let $\Psi \in \mathrm{Hom}_R(R\otimes_A M, N)$.
We need to see that
$$\Phi(\underbrace{(a\otimes b)\cdot}_{\text{LHS action}}\Psi) = \underbrace{(a\otimes b)\cdot}_{\text{RHS action}}\Phi(\Psi).$$

Plug in m to both sides.

$\Phi((a\otimes b)\cdot\Psi)(m) = (a\otimes b)\Psi(1\otimes m) = \Psi(a\otimes bm)$
$((a\otimes b)\cdot\Phi(\Psi))(m) = a(\Phi(\Psi)(bm)) = a\Psi(1\otimes bm)$.
Since Ψ is R-linear, using R-mod structure we have
$\Psi(a\otimes bm) = \Psi(a(1\otimes bm)) = a\Psi(1\otimes bm)$.
Thus, the module structures are compatible.

Now we check the map is bijective. By Hom-$\otimes$ adjunction as A-modules, we have

$$\mathrm{Hom}_A(R\otimes_A M, N) \simeq \mathrm{Bil}_A(R\times M, N) \simeq \mathrm{Hom}_A(R, \mathrm{Hom}_A(M,N))$$
$$\psi \longmapsto \psi\circ((r,m)\mapsto r\otimes m) \longmapsto (r\mapsto (\psi(r\otimes -)))$$

We claim that this restricts to the iso we want; i.e.,
ψ is R-linear $\iff$ $r\mapsto \psi(r\otimes -)\in \mathrm{Hom}_A(R,\mathrm{Hom}_A(M,N))$ is R-linear.
But ψ R-linear $\iff$ $\psi(r\otimes m) = r\psi(1\otimes m)$ for all $r\in R, m\in M$,
and $r\mapsto \psi(r\otimes -)$ R-linear $\overset{\text{(apply to } m)}{\iff}$ $\psi(r\otimes m) = r\psi(1\otimes m)$ $\forall$.
Thus, the iso above restricts to an iso:

$$\mathrm{Hom}_R(R\otimes_A M, N) \xrightarrow{\sim} \mathrm{Hom}_R(R, \mathrm{Hom}_A(M,N)) \xrightarrow[\text{evaluate at } 1]{\sim} \mathrm{Hom}_A(M,N)$$
$$\psi \longmapsto (r\mapsto \psi(r\otimes -)) \longmapsto \psi(1\otimes -),$$

and this is just the map Φ! ☐

We will use these module structures to characterize differential operators. It begins with an observation:

Lem: Let $\alpha\in \mathrm{Hom}_A(M,N)$ and $f\in R$. Then $[\alpha, f] = (1\otimes f - f\otimes 1)\cdot\alpha$ under the $R\otimes_A R$-module structure specified.

pf: We have $(1\otimes f)\cdot\alpha(-) = \alpha(f-)$, so $(1\otimes f)\cdot\alpha = \alpha f$
and $(f\otimes 1)\cdot\alpha(-) = f\alpha(-)$, so $(f\otimes 1)\alpha = f\alpha$. ☐

Let $A \to R$ commut. rings. Then there is a ring homomorphism $R\otimes_A R \xrightarrow{\mu} R$ given by $\mu(a\otimes b) = ab$. We define

$$\Delta_{R|A} = \ker(R\otimes_A R \xrightarrow{\mu} R).$$

We observe that $\Delta_{R|A}$ is generated by elements of the form $1\otimes f - f\otimes 1$, $f\in R$: Each such element is in the kernel, and there is an iso $R\otimes_A R/(\{1\otimes f - f\otimes 1\}) \cong R$, since every element of $R\otimes_A R$ is equivalent modulo $(\{1\otimes f - f\otimes 1\})$ to an element of the form $r\otimes 1$, and there is an R-linear inverse $r \mapsto r\otimes 1 + (\{1\otimes f - f\otimes 1\})$.

Moreover, if $R = A[f_1,\dots,f_t]$, then

$$\Delta_{R|A} = (1\otimes f_1 - f_1\otimes 1, \dots, 1\otimes f_t - f_t\otimes 1).$$

This follows from the identity

$$1\otimes fg - fg\otimes 1 = (1\otimes f - f\otimes 1)(1\otimes g) + (1\otimes g - g\otimes 1)(f\otimes 1).$$

Prop: The collection $D^i_{R|A}(M,N) \subseteq \mathrm{Hom}_A(M,N)$ is the $R\otimes_A R$-submodule of $\mathrm{Hom}_A(M,N)$ annihilated by $\Delta^{i+1}_{R|A}$.

Pf: By induction on i. For $i=0$, $\delta \in D^0_{R|A}(M,N)$ if and only if $[\delta,\bar{r}]=0$ for each $r\in R$, i.e., iff each $[\cdot,\bar{r}]$ operation annihilates δ. This is equivalent to each element $(1\otimes r - r\otimes 1)$ annihilating δ, which is equivalent to $\Delta_{R|A}$ annihilating δ.

The inductive step is similar: for the same reason, we have $\delta \in D^i_{R|A}(M,N) \iff \Delta_{R|A}\cdot\delta \in D^{i-1}_{R|A}(M,N)$
$\overset{IH}{\iff} \Delta_{R|A}\cdot\Delta^{i}_{R|A}\delta = 0 \iff \Delta^{i+1}_{R|A}\cdot\delta = 0.$

Def: ~~Set~~ $P^i_{R|A} := (R\otimes_A R)/\Delta^{i+1}_{R|A}$
and $P^i_{R|A}(M) := (R\otimes_A M)/\Delta^{i+1}_{R|A}\cdot(R\otimes_A M)$.
These are $(R\otimes_A R)$-modules, and we view them as R-modules by the action on the left copy of $R\otimes_A R$ or $R\otimes_A M$. I.e.,
$r\cdot(a\otimes m + \Delta^{i+1}_{R|A}(R\otimes_A M)) = ra\otimes m + \Delta^{i+1}_{R|A}(R\otimes_A M)$.
We call $P^i_{R|A}$ the <u>module of principal parts</u> of order i.

<u>Thm:</u> There is an isomorphism of $R\otimes_A R$-modules.
$\mathrm{Hom}_R(P^i_{R|A}(M),N) \xrightarrow{\sim} D^i_{R|A}(M,N)$.

This iso. is given by the composition
$\mathrm{Hom}_R(P^i_{R|A}(M),N) \xrightarrow{\pi^*} \mathrm{Hom}_R(R\otimes_A M,N) \xrightarrow{d_M^*} D^i_{R|A}(M,N)$,
where π^* is the map $\mathrm{Hom}(\pi,N)$ induced by the quotient $R\otimes_A M \xrightarrow{\pi} P^i_{R|A}(M)$, and d_M^* is the (restriction of) the map Φ that is precomposition by d_M.

pf: Consider the SES of $R\otimes_A R$-mods or R-mods

$$0 \to \Delta^{i+1}_{R|A}(R\otimes M) \to R\otimes M \xrightarrow{\pi} P^i(M) \to 0.$$

By left-exactness ~~of~~ $\mathrm{Hom}_R(-, N)$, we have

$$0 \to \mathrm{Hom}_R(P^i(M), N) \xrightarrow{\pi^*} \mathrm{Hom}_R(R\otimes M, N) \to \mathrm{Hom}_R(\Delta^{i+1}(R\otimes M), N)$$

exact, which means π^* is injective, and its image is the set of maps ~~killed~~ that are zero on $\Delta^{i+1}(R\otimes M)$. Since we view $\mathrm{Hom}_R(R\otimes_A M, N)$ as a $(R\otimes R)$-module via the action on the source, φ is zero on $\Delta^{i+1}(R\otimes M) \iff \Delta^{i+1}\cdot\varphi = 0$ in $\mathrm{Hom}(R\otimes M, N)$. Thus, $d_M \circ \pi^*$ induces an isomorphism between $\mathrm{Hom}_R(P^i(M), N)$ and $\mathrm{ann}_{\mathrm{Hom}_A(M,N)}(\Delta^{i+1}_{R|A}) = D^i_{R|A}$. $\square$