III. Differential operators in general

Def: Let $A \to R$ be a homomorphism of commutative rings. Let M and N be two R-modules. The differential operators of order i from M to N are defined inductively in i as follows:

$$D^0_{R|A}(M,N) = \mathrm{Hom}_R(M,N)$$

$$D^i_{R|A}(M,N) = \left\{ \delta \in \mathrm{Hom}_A(M,N) \;\middle|\; \begin{array}{l} \delta \circ \bar{f}_M - \bar{f}_N \circ \delta \in D^{i-1}_{R|A}(M,N) \\ \text{for all } f \in R \end{array} \right\}.$$

We define $D_{R|A}(M,N) = \bigcup_{i \in \mathbb{N}} D^i_{R|A}(M,N)$.

Obs: A function $\alpha \in \mathrm{Hom}_A(M,N)$ is R-linear (i.e., $\alpha \in \mathrm{Hom}_R(M,N)$) if and only if $\delta \circ \bar{f}_M = \bar{f}_N \circ \delta$ for all $f \in R$. Thus, we get the same notion if we set
$D^{-1}_{R|A}(M,N) = 0$
and use same inductive step.

Notation: Given $\alpha, \beta \in \mathrm{End}_A(M)$ some A-module M, we write $[\alpha, \beta] := \alpha \circ \beta - \beta \circ \alpha \in \mathrm{End}_A(M)$.
We also abuse this notation, e.g., by writing $[\alpha, f]$ for $\alpha \circ \bar{f}_M - \bar{f}_N \circ \alpha$ for $\alpha \in \mathrm{Hom}_A(M,N)$.

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and using the $\bar{R}$-module structure above, this is an isomorphism. In this way, we can think of $D_{R|k}$ as close to a polynomial ring in $2n$ variables.

(4a) Properties of the bracket

For α, β, γ homomorphisms of modules or "$\bar{f}$", the following hold whenever defined.

i) $[\alpha, \beta+\gamma] = [\alpha,\beta] + [\alpha,\gamma]$
   and $[\alpha+\beta,\gamma] = [\alpha,\gamma] + [\beta,\gamma]$

ii) $[\bar{a}\alpha, \beta] = [\alpha, \bar{a}\beta] = \bar{a}[\alpha,\beta]$ if $a \in A$ & α, β A-linear

iii) $[\alpha,\beta] = -[\beta,\alpha]$

iv) $[\alpha\beta,\gamma] = \alpha[\beta,\gamma] + [\alpha,\gamma]\beta$
   and $[\alpha,\beta\gamma] = [\alpha,\beta]\gamma + \beta[\alpha,\gamma]$

v) $[[\alpha,\beta],\gamma] + [[\beta,\gamma],\alpha] + [[\gamma,\alpha],\beta] = 0.$

pf: i) $[\alpha, \beta+\gamma] = \alpha(\beta+\gamma) - (\beta+\gamma)\alpha = \alpha\beta + \alpha\gamma - \beta\alpha - \gamma\alpha$
$= \alpha\beta - \beta\alpha + \alpha\gamma - \gamma\alpha = [\alpha,\beta] + [\alpha,\gamma]$ & similarly.

etc.

v). $LHS = (\alpha\beta - \beta\alpha)\gamma - \gamma(\alpha\beta - \beta\alpha) + (\beta\gamma - \gamma\beta)\alpha - \alpha(\beta\gamma - \gamma\beta)$
$+ (\gamma\alpha - \alpha\gamma)\beta - \beta(\gamma\alpha - \alpha\gamma)$. all cancels.

5

Lemma: $\bar{r} \in R$, $\alpha \in D^i_{R|A} \Rightarrow \bar{r}\alpha \in D^i_{R|A}$. $i=0$ clear

pf: $[\bar{r}\alpha, \bar{f}] = \bar{r}[\alpha, \bar{f}] + [\bar{r}, \bar{f}]\alpha = \bar{r}[\alpha, \bar{f}]$
and it follows from induction.
Similarly for $\alpha\bar{r}$.

Prop: Let $R = A[f_1, \ldots, f_t]$; i.e., R is gen by $\{f_1, \ldots, f_t\}$ as an A-algebra. Then $\delta \in \mathrm{End}_A(R)$ is in $D^i_{R|A}$
$\iff [\delta, \bar{f}_j] \in D^{i-1}_{R|A}$ $j = 1, \ldots, t$.

pf: By A-linearity, it suffices to show that the hypothesis implies $[\delta, \bar{f}_1^{a_1} \cdots \bar{f}_t^{a_t}] \in D^{i-1}_{R|A}$. Write $f_1^{a_1} \cdots f_t^{a_t} = f_i \cdot \mu$, μ of "lower degree" in the f's. Then
$[\delta, \bar{f}_i \bar{\mu}] = [\delta, \bar{f}_i]\bar{\mu} + \bar{f}_i[\delta, \bar{\mu}] \in D^{i-1}_{R|A}$ inductively. ☐

We now generalize the lemma above.

Prop: Let $A \to R$ comm. rings, and L, M, N R-mods. Then $\alpha \in D^i_{R|A}(M, N)$, $\beta \in D^j_{R|A}(L, M)$
$\Rightarrow \alpha \circ \beta \in D^{i+j}_{R|A}(L, N)$. $i+j=0$ clear

pf: $[\alpha \circ \beta, \bar{f}] = \alpha[\beta, \bar{f}] + [\alpha, \bar{f}]\beta$
and apply induction on $i+j$.

Consequence: If $A \to R$ comm. rings, then $D_{R|A}$ is a ring under composition. If
$:= D_{R|A}(R, R)$
M is an R-module, $D_{R|A}(M, M)$ is a ring.

pf: As subsets of $\mathrm{End}_A(R)$ or $\mathrm{End}_A(M)$, we just need to check these are closed under composition & subtraction (& have 1). ⊠

Moreover, these are filtered rings: $D^i_{R|A} \cdot D^j_{R|A} \subseteq D^{i+j}_{R|A}$, and likewise with $D(M,M)$.

We have an isomorphism $R \cong D^0_{R|A}$ of rings; we will also write $\overline{R}$ for $D_{R|A}$. The image of A is contained in the center of $D_{R|A}$ by definition.

Prop: Let $R = A[f_1, \dots, f_t]$, and $\alpha, \beta \in D^i_{R|A}$. Then $\alpha = \beta \iff \alpha(f_1^{c_1} \cdots f_t^{c_t}) = \beta(f_1^{c_1} \cdots f_t^{c_t})$ for all $c_1 + \dots + c_t \leq i$.

pf: Considering $\delta = \alpha - \beta$, it suffices to show $\delta \in D^i_{R|A}$ is zero $\iff$ it is zero on every "monomial" as above. By induction on i; WLOG $\delta \notin D^{i-1}$. (Note that $i=0$ case is clear: $\overline{r} = \overline{s} \iff r = \overline{r}(1) = \overline{s}(1) = s$). Then, $\exists j$ s.t. $[\delta, f_j] \neq 0$, since $\delta \in D^0_{R|A}$ otherwise, and by IH, $\exists f_1^{d_1} \cdots f_t^{d_t}$ with $d_1 + \dots + d_t \leq i-1$ and $[\delta, f_j](f_1^{d_1} \cdots f_t^{d_t}) \neq 0$. But this is $\delta(f_1^{d_1} \cdots f_j^{d_j+1} \cdots f_t^{d_t}) - f_j\, \delta(f_1^{d_1} \cdots f_t^{d_t})$, so either $(\text{———}''\text{———})$ or $(\text{——}''\text{——})$ is nonzero. ⊠

Theorem: The two notions of differential operators on $R = K[x_1, \ldots, x_n]$, K field of char 0 agree. Namely
$D_{R|K}$ by inductive definition $= K\langle \bar{x}_1, \ldots, \bar{x}_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle$.

pf: Let us write $\tilde{D}^i_{R|K}$ for $\bigoplus_{\alpha_1 + \cdots + \alpha_n \leq i} \bar{R} \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n}$.
First, we show $\tilde{D}^i_{R|K} \subseteq D^i_{R|K}$. For $i=0$, this is clear.
Note that $[\frac{\partial}{\partial x_i}, \bar{x}_j] = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$

By a proposition from earlier, to check membership in $D^{\alpha}_{R|K}$, suffices to check $[\,,\,]$ with any generator of R, so $\frac{\partial}{\partial x_j} \in D^1_{R|K}$ for each j.
Then $\frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n} \in D^{|\alpha|}_{R|K}$ (by proposition on compositions), and $\bar{R} \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n} \subseteq D^{|\alpha|}_{R|K}$, so $\tilde{D}^i_{R|K} \subseteq D^i_{R|K}$.

For the other containment, we claim that the restriction map $\tilde{D}^i_{R|K} \longrightarrow \mathrm{Hom}_K(R_{\leq i}, R)$ is an isomorphism. ~~that~~ Since $\tilde{D}^i_{R|K} \subseteq D^i_{R|K}$, the previous prop. shows that this is injective. To see surjective, we ~~need~~ to see that ~~[illegible]~~ for any $\{S_\alpha\}_{\alpha_1 + \cdots + \alpha_n \leq i} \subseteq R$, $\exists\, \delta \in \tilde{D}^i_{R|K}$ with $\delta(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = S_\alpha$.
Recall ~~Note~~ that $(\frac{\partial}{\partial x_1}^{\beta_1} \cdots \frac{\partial}{\partial x_n}^{\beta_n})(x_1^{\alpha_1} \cdots x^{\alpha_n}) = \begin{cases} \beta_1! \cdots \beta_n! & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha_1 + \cdots + \alpha_n \leq \beta_1 + \cdots + \beta_n \text{ and } \alpha \neq \beta \end{cases}$

Given $\{S_\alpha\}_{|\alpha| \leq i}$, inductively we can find $\delta' \in \tilde{D}^{i-1}_{R|K}$ with $\delta'(\underline{x}^{\alpha}) = S_\alpha$ for $|\alpha| \leq i-1$; then take $\delta = \delta' + \sum_{|\alpha| = i} \left[\frac{S_\alpha - \delta'(\underline{x}^{\alpha})}{\alpha_1! \cdots \alpha_n!} \left(\frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n} \right) \right]$.

Now, for any $\delta \in D^i_{R|K}$, $\exists \tilde{\delta} \in \tilde{D}^i_{R|K}$ such that $\delta|_{R_{\leq i}} = \tilde{\delta}|_{R_{\leq i}}$. By the previous proposition again, we have $\delta = \tilde{\delta}$. Thus, $\tilde{D}^i_{R|K} = D^i_{R|K}$. ☐

We record a fact we proved along the way:

Cor: If K field of char 0, $R = K[x_1, \dots, x_n]$, then the restriction map $D^i_{R|K} \longrightarrow \mathrm{Hom}_K(R_{\leq i}, R)$ is an isomorphism.

Theorem: Let $A \to R$ be a map of rings. Then the associated graded ring of the order filtration of $D_{R|A}$ ($\mathrm{gr}(D^{\bullet}_{R|A})$) is commutative.

pf: We want to show that $\alpha \in D^i_{R|A}, \beta \in D^j_{R|A}$ $\Rightarrow$ $\alpha\beta + D^{i+j-1}_{R|A} = \beta\alpha + D^{i+j-1}_{R|A}$, or equivalently that $[\alpha, \beta] \in D^{i+j-1}_{R|A}$. We induce on $i+j$. Let $f \in R$. Then

$$[[\alpha,\beta], f] = -[[\beta, f], \alpha] - [[f, \alpha], \beta]$$
$$= [[\alpha, f], \beta] - [[\beta, f], \alpha].$$

We have $[\alpha, f] \in D^{i-1}_{R|A}$ and $[\beta, f] \in D^{j-1}_{R|A}$, so IH applies, and RHS $\in D^{i+j-2}_{R|A}$. Thus, $[\alpha, \beta] \in D^{i+j-1}_{R|A}$, as required. ☐