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A bit more about differential operators on the polynomial ring.

K field of char 0

$R = K[x_1, \dots, x_n]$ poly ring

$D_{R|K} = K\langle \overline{x_1}, \dots, \overline{x_n}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle \subseteq \mathrm{End}_K(R)$

relations:

$$\overline{x_i}\,\overline{x_j} = \overline{x_j}\,\overline{x_i} \quad i \neq j \qquad [\overline{x_i}, \overline{x_j}] = 0$$

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} \quad i \neq j \quad \rightsquigarrow \quad \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

$$\overline{x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\overline{x_i} \quad i \neq j \qquad \left[\overline{x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

$$\overline{x_i}\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i}\overline{x_i} - \overline{1} \qquad \left[\overline{x_i}, \frac{\partial}{\partial x_i}\right] = -\overline{1}.$$

where $[\alpha, \beta] := \alpha\beta - \beta\alpha$.

Want to use these relations to write any diff'l operator in a standard form / recognize when two operators are same or different.

Lemma: i) $\left(\frac{\partial}{\partial x_i}\right)^a \overline{x_i} = \overline{x_i}\left(\frac{\partial}{\partial x_i}\right)^a + a\left(\frac{\partial}{\partial x_i}\right)^{a-1}$

ii) $\left(\frac{\partial}{\partial x_i}\right)^a \overline{x_i}^{\,b} = \sum_{k=0}^{\min\{a,b\}} k!\binom{a}{k}\binom{b}{k}\overline{x_i}^{\,b-k}\left(\frac{\partial}{\partial x_i}\right)^{a-k}$

pf: i) By induction on a, with $a=1$ already done.

Ind. step: $\left(\frac{\partial}{\partial x_i}\right)^a \overline{x_i} = \left(\frac{\partial}{\partial x_i}\right)\left(\frac{\partial}{\partial x_i}\right)^{a-1}\overline{x_i} \overset{IH}{=} \frac{\partial}{\partial x_i}\left(\overline{x_i}\left(\frac{\partial}{\partial x_i}\right)^{a-1} + (a-1)\left(\frac{\partial}{\partial x_i}\right)^{a-2}\right)$

$= \left(\overline{x_i}\frac{\partial}{\partial x_i} + \overline{1}\right)\left(\frac{\partial}{\partial x_i}\right)^{a-1} + (a-1)\left(\frac{\partial}{\partial x_i}\right)^{a-1} = \overline{x_i}\left(\frac{\partial}{\partial x_i}\right)^a + a\left(\frac{\partial}{\partial x_i}\right)^{a-1}.$

ii) Similar, but messy.

Rather than the precise form of (ii), we will mostly care about this as saying we can switch the order and write as same ~~term~~ product "reversed" (note first coeff is 1) plus smaller terms.

<u>proposition</u>: Any element $\delta \in D_{R|k}$ can be written as
$$\delta = \sum_{\alpha_1, \dots, \alpha_n} \bar{r}_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$
for some $\bar{r}_\alpha$'s $\in R$.

That is, $D_{R|k}$ is generated by $\{(\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}\}$ as an left $\bar{R}$-module, where $\bar{R}$ is the image of R in $D_{R|k}$.

<u>prf</u>: Using the commutation relations, we can express any element as a sum of products of the form $\left(\bar{x}_1^{a_{11}} \frac{\partial}{\partial x_1}^{b_{11}} \bar{x}_1^{a_{12}} \frac{\partial}{\partial x_1}^{b_{12}} \cdots \bar{x}_1^{a_{1j}} \frac{\partial}{\partial x_1}^{b_{1j}}\right)$ and similar terms in other indices. Apply lemma to "straighten out" $\frac{\partial}{\partial x_1}^{b_{1j-1}} \bar{x}_1^{a_{1j}}$ as a sum of products ~~w~~ with $\bar{x}$'s before $\frac{\partial}{\partial x}$'s. Inductively, we obtain elements of desired form.

<u>Theorem</u>: The expressions in the previous proposition are unique. That is, $\{(\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}\}$ is a free basis for $D_{R|k}$ as a (left) $\bar{R}$-module.

<u>prf</u>: We need to show that
$\delta = \sum \bar{r}_\alpha \frac{\partial}{\partial x_1}^{\alpha_1} \cdots \frac{\partial}{\partial x_n}^{\alpha_n} = 0$ implies each $\bar{r}_\alpha$ is zero.
Given such a relation, pick a tuple β with $\bar{r}_\beta$ nonzero above, and $\beta_1 + \cdots + \beta_n$ minimal among such tuples. Compute $\delta(x_1^{\beta_1} \cdots x_n^{\beta_n})$: we know this is 0.
If $\alpha_1 + \cdots + \alpha_n \geq \beta_1 + \cdots + \beta_n$, then $\alpha = \beta$ or else $\alpha_i > \beta_i$ for some i. Thus, if $\bar{r}_\alpha \neq 0$ & $\alpha \neq \beta$, we have
$$\left(\bar{r}_\alpha \frac{\partial}{\partial x_1}^{\alpha_1} \cdots \frac{\partial}{\partial x_n}^{\alpha_n}\right)(x_1^{\beta_1} \cdots x_n^{\beta_n}) = 0, \quad \text{so}$$
$$\begin{aligned}\delta(x_1^{\beta_1} \cdots x_n^{\beta_n}) &= \sum_\alpha \bar{r}_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} (x_1^{\beta_1} \cdots x_n^{\beta_n}) \\ &= \bar{r}_\beta \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n} (x_1^{\beta_1} \cdots x_n^{\beta_n}) \\ &= \beta_1! \cdots \beta_n! \, \bar{r}_\beta,\end{aligned}$$
contradicting a choice of nonzero r_β. $\square$

$\nexists$ $\bar{R}$-basis

<u>Rmk</u>: char $k = 0$ was used in an important way here. In char p, have $(\frac{\partial}{\partial x_i})^p = 0$!

Order filtration

We define an ascending filtration on $D_{R|k}$ by $D^i_{R|k} = \bigoplus_{\alpha_1+\dots+\alpha_n \le i} \bar{R} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$;

these are the elements of order at most i.

We observe that this multiplicative

$$D^i_{R|k} \cdot D^j_{R|k} \subseteq D^{i+j}_{R|k} \quad \text{for each } i,j.$$

It suffices to check for "monomials"

$$\left(\bar{x}_1^{a_1} \cdots \bar{x}_n^{a_n} \frac{\partial}{\partial x_1}^{b_1} \cdots \frac{\partial}{\partial x_n}^{b_n}\right)\left(\bar{x}_1^{c_1} \cdots \bar{x}_n^{c_n} \frac{\partial}{\partial x_1}^{d_1} \cdots \frac{\partial}{\partial x_n}^{d_n}\right)$$

$$b_1 + \dots + b_n \le i \qquad d_1 + \dots + d_n \le j.$$

Can write $\frac{\partial}{\partial x_1}^{b_1} \bar{x}_1^{c_1}$ as a sum in standard form with ~~elements~~ partials to the $\le b_1$; similarly for each i.

Go to (A1) ☆

Grading on $D_{R|k}$

An element δ of $D_{R|k}$ is homogeneous of degree d if $\delta(R_i) \subseteq R_{i+d}$ for each i.

The ring $D_{R|k}$ is graded in this way: any element is uniquely a sum of homogeneous elements, and products of homog. elements are homog. of degree equal to the sum of the degrees. E.g., $\deg(\frac{\partial}{\partial x_i}) = -1$, $\deg(\bar{x}_2^3 \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} + \bar{x}_3) = 1$;

(A1). Associated graded of order filtration

Given a multiplicative ascending filtration

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

of a ring T, the associated graded ring of the filtration is

$$gr_{F_\bullet}(T) = \bigoplus_{i \geq 0} F_i / F_{i-1} \qquad (F_{-1} := 0)$$

as graded abelian groups, with multiplication

$$(\underset{F_i}{f} + F_{i-1})(\underset{F_j}{g} + F_{j-1}) = (\underset{F_{i+j}}{fg} + F_{i+j-1}) ;$$

this is well-defined by multiplicative hypothesis.

We compute this for the ring of diff. ops. on poly ring in char 0, with order filtration. We have

$$gr^{ord}(D_{R|K})_i = \frac{D^i_{R|K}}{D^{i-1}_{R|K}} \cong \bigoplus_{\alpha_1+\dots+\alpha_n=i} \overline{R} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

or just check gens commute.

We observe that

$$\left(\overline{r} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} + D^{i-1}_{R|K}\right)\left(\overline{s} \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n} + D^{j-1}_{R|K}\right)$$

$$= \overline{r}\,\overline{s} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1+\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n+\beta_n} + D^{i+j-1}_{R|K} :$$

use that $\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} x_1^{c_1} = x_1^{c_1} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}$ + lower order terms...

Thus, $gr^{ord}(D_{R|K})$ is commutative.

There is a ring homomorphism (K-alg. homom)

$$K[x_1, \dots, x_n, y_1, \dots, y_n] \longrightarrow gr^{ord}(D_{R|K})$$

$$x_i \longmapsto \overline{x_i}$$

$$y_i \longmapsto \frac{\partial}{\partial x_i} + D^0_{R|K},$$

and using the $\bar{R}$-module structure above, this is an isomorphism. In this way, we can think of $D_{R|k}$ as close to a polynomial ring in $2n$ variables.

(4a) <u>Properties of the bracket</u>

For α, β, γ homomorphisms of modules or "$\mathfrak{F}$", the following hold whenever defined.

i) $[\alpha, \beta+\gamma] = [\alpha,\beta] + [\alpha,\gamma]$ and $[\alpha+\beta,\gamma] = [\alpha,\gamma] + [\beta,\gamma]$

ii) $[\bar{a}\alpha, \beta] = [\alpha, \bar{a}\beta] = \bar{a}[\alpha,\beta]$ if $a \in A$ & α,β A-linear

iii) $[\alpha,\beta] = -[\beta,\alpha]$

iv) $[\alpha\beta,\gamma] = \alpha[\beta,\gamma] + [\alpha,\gamma]\beta$ and $[\alpha,\beta\gamma] = [\alpha,\beta]\gamma + \beta[\alpha,\gamma]$

v) $[[\alpha,\beta],\gamma] + [[\beta,\gamma],\alpha] + [[\gamma,\alpha],\beta] = 0$.

<u>prf:</u> i) $[\alpha, \beta+\gamma] = \alpha(\beta+\gamma) - (\beta+\gamma)\alpha = \alpha\beta + \alpha\gamma - \beta\alpha - \gamma\alpha$
$= \alpha\beta - \beta\alpha + \alpha\gamma - \gamma\alpha = [\alpha,\beta] + [\alpha,\gamma]$ & similarly.

etc.